Note

Every Finite Distributive Lattice Is a Set of Stable Matchings

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Suppose we have n men and n women. Each of the $2n$ people has a linear preference ordering on those of the opposite sex. We are interested in matchings to form n couples. A matching is stable if we cannot find a woman in one couple and a man in another who would prefer each other to their present partners.

Stable matchings were first defined by Gale and Shapley [1], who showed that for any preference orderings a stable matching always exists. In general, there will be several stable matchings. For example, if all the men happen to have different first preferences, giving each man his first choice will be stable, regardless of the women's preferences. Similarly, giving each woman her first choice (if possible) will be stable.

Conway [2, p. 87–92] defines a partial ordering on the set of stable matchings as follows: one matching is $\geqslant$ another if every man is at least as happy with his partner in the first matching as with his partner in the second. He shows that the set of stable matchings is always a finite distributive lattice. Knuth [2, p. 92] asks whether every finite distributive lattice can occur as the set of stable matchings generated by some set of men and women. We show this is the case.

We will require a preliminary lemma about finite distributive lattices. If L is such a lattice and $x \in L$ we define a new lattice L^x by making a copy of all the elements $\leqslant x$ and putting the copies immediately below the originals. More formally, if V is disjoint from L with one element $v_y \in V$ for each $y \leqslant x$, then L^x is the partial ordering on $L \cup V$ given by (i) if $w, z \in L$ then $w \geqslant z$ in L^x iff $w \geqslant z$ in L; (ii) if $w \in L$ and $v_z \in V$, then $w \geqslant v_z$ in L^x iff $w \geqslant z$ in L; (iii) $v_w \geqslant v_z$ iff $w \geqslant z$; (iv) $v_w \not\geqslant z$ for any w, z.

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LEMMA 1. *If a set S of lattices includes a one-element lattice and includes a lattice isomorphic to L^x for every $L \in S$, $x \in L$ then every finite distributive lattice is isomorphic to a lattice in S.*

Proof. Let M be a finite distributive lattice. We argue by induction on the size of M. If M has one element the result is immediate. Otherwise let z be the smallest member of M which is not the meet of two members different from z. Let w be the meet of all members $>z$. $N = \{y \mid y \not\leqslant z\}$ is a distributive sublattice in which meets and joins are preserved. It can be shown that if $y \leqslant z$ then $y = z \wedge u$ for some $u \in N$. Moreover, if $u_1 \wedge z = u_2 \wedge z \neq z$ then $u_1 \wedge w = u_2 \wedge w$. Hence the mapping from u to $z \wedge u$ is an isomorphism from $\{u \mid u \leqslant w$ and $u \not\leqslant z\}$ to $\{u \mid u \leqslant z\}$. Therefore M is isomorphic to N^w. By induction hypothesis, N is isomorphic to a lattice in S, so M is isomorphic to a lattice in S.   Q.E.D.

LEMMA 2. *If we have n men and n women for whom the set of stable matchings is L, and $x \in L$, then there are 2n men and 2n women with preferences such that the set of stable matchings is L^x.*

Proof. Let L be the set of stable matchings possible for women $w_1,..., w_n$ and men $m_1,..., m_n$. Let x denote the matching in which m_i gets w_i for all i and assume this is stable, i.e., a member of L. Then we will show the set of stable matchings for the $2n$ men $m_1,..., m_n$; $m'_1,..., m'_n$ and women $w_1,..., w_n$; $w'_1,..., w'_n$ with the following preferences is isomorphic to L^x:

m_i: Use the original preferences of m_i in the n-couple situation for all women strictly preferred to (above) w_i. Replace w_i by w'_i. After w'_i put w'_{i+1} and finish the ordering arbitrarily.

m'_i: First choice w'_i, followed by w_i and the original preferences of m_i below w_i. Finish arbitrarily.

w_i: In the original preference ordering put m'_j directly below m_j for all j. For example, if m_5 is first choice and m_3 second in the n-couple situation, then the new ordering is $m_5 m'_5 m_3 m'_3$.

w'_i: First choice is m_{i-1}. Second choice is m_i, followed by m'_i. Finish arbitrarily.

In this definition all arithmetic is modulo n. We illustrate with an example after the proof.

Proof. We begin by observing that in any stable $2n$-couple matching with these preferences: (1) If for some i, m_i gets w'_{i+1} then w'_i (preferred by m_i) must get m_{i-1}, hence m_i must get w'_{i+1} for all i. (2) w'_i is the first choice of m'_i, hence w'_i must get either m_{i-1} (and (1) applies), or m_i, or m'_i. (3) m_i must get somebody at least as good as w'_{i+1}. (4) If m_i does not get w'_i or w'_{i+1}, then m'_i gets w'_i. (5) If m_i prefers w_j to w'_i, m'_i does not get w_j. (w_j

prefers m_i to m_i'. If m_i' got w_j, (4) would imply m_i gets w_i' or w_{i+1}' so m_i and w_j would be happier together.)

These observations imply that nobody gets assigned to the arbitrary part of his or her ordering. Further if we are given a stable matching for the $2n$ couples we obtain a stable matching for the n-couple problem (i.e., a member of L) by giving each w_i her partner in the $2n$-couple problem, deleting primes where necessary. Conversely if $y \in L$, there is a corresponding stable matching for the $2n$-couple situation in which m_i is replaced by m_i' iff m_i gets w_i or somebody worse in y. If $y \leqslant x$ there are two $2n$-couple matches corresponding to y—one in which each m_i gets w_i', and one in which each m_i gets w_{i+1}'. Those matches in which each m_i gets m_{i+1}' correspond to V in the definition of L^x. Q.E.D.

The construction in Lemma 2 also works when x is a matching that is not stable. We have not considered the implications of this.

EXAMPLE. The eight people with preferences given below have stable matching corresponding to a 4-element lattice: (A) m_1 gets w_2, m_2 gets w_1, m_3 gets w_3, m_4 gets w_4 (abbreviated (2134)) (B) (1243) (C) (1234) (D) (2143). (A) is the maximal element of L, (B) minimal.

$$m_1 \; m_2 \; m_3 \; m_4 \qquad w_1 \; w_2 \; w_3 \; w_4$$
$$w_2 \; w_1 \; w_3 \; w_4 \qquad m_1 \; m_2 \; m_4 \; m_3$$
$$w_1 \; w_2 \; w_4 \; w_3 \qquad m_2 \; m_1 \; m_3 \; m_4 \qquad (\cdots = \text{arbitrary})$$
$$\cdots \qquad\qquad \cdots$$

$L^{(1234)}$ is a 6-element lattice generated by the preferences:

$$m_1 \; m_2 \; m_3 \; m_4 \; m_1' \; m_2' \; m_3' \; m_4' \qquad w_1 \; w_2 \; w_3 \; w_4 \; w_1' \; w_2' \; w_3' \; w_4'$$
$$w_2 \; w_1 \; w_3' \; w_4' \; w_1' \; w_2' \; w_3' \; w_4 \qquad m_1 \; m_2 \; m_4 \; m_3 \; m_4 \; m_1 \; m_2 \; m_3$$
$$w_1' \; w_2' \; w_4' \; w_1' \; w_1 \; w_2 \; w_3 \; w_4 \qquad m_1' \; m_2' \; m_4' \; m_3' \; m_1 \; m_2 \; m_3 \; m_4$$
$$w_2' \; w_3' \qquad\qquad\qquad w_4 \; w_3 \qquad m_2 \; m_1 \; m_3 \; m_4 \; m_1' \; m_2' \; m_3' \; m_4'$$
$$\cdots \qquad\qquad\qquad\qquad\qquad m_2' \; m_3' \; m_3' \; m_4'$$
$$\cdots$$

The stable matchings are $(213'4'1'234)$, $(1'2'3'4'1234)$, $(1'2'3'4'1243)$, $(213'4'1'243)$, $(2'3'4'1'1234)$, and $(2'3'4'1'1243)$. The last two are members of V.

The result announced in the title follows immediately from Lemmas 1 and 2. Examination of the proofs yields

THEOREM. *If L is a finite distributive lattice with n elements there are*

sets of men and women such that the lattice of their stable matchings is isomorphic to L. The number of men (= number of women) is $\leqslant 2^n$.

The construction we have given does not use the smallest number of people needed to represent a given lattice. The 6-element lattice in the example can be represented using ten people as follows

$$
\begin{array}{ccccc}
m_1 \ m_2 \ m_3 \ m_4 \ m_5 & \quad & w_1 \ w_2 \ w_3 \ w_4 \ w_5 \\
w_1 \ w_2 \ w_3 \ w_4 \ w_5 & \quad & m_2 \ m_3 \ m_1 \ m_5 \ m_4 \\
w_3 \ w_3 \ w_2 \ w_5 \ w_4 & \quad & m_1 \ m_2 \ m_2 \ m_4 \ m_5 \\
w_1 & \quad & m_3 \\
\cdots & \quad & \cdots
\end{array}
$$

The stable matches are (12345), (12354), (13245), (13254), (31245), and (31254). However, it is not possible in general to go from L to L^x by adding only one additional couple.

The structure of the set of matches is clearly reminiscent of the representation of a permutation by cycles. I wish to thank Alvin Roth, whose recent work [3] motivated this note.

REFERENCES

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2. D. KNUTH, "Marriage Stables," Montreal Univ. Press, Montreal, 1976.
3. A. ROTH, The economics of matching: Stability and incentives, *Math. Oper. Res.* **7** (1982), 617–628.