Stochastic Processes 2026–27

Problem Set 12: The Fokker–Planck Equation

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Instructions. For the Itô diffusion

$$dX_t = b(X_t, t)\,dt + \sigma(X_t, t)\,dW_t,$$

you may use the Fokker–Planck equation

$$\partial_t p = -\partial_x(bp) + \frac{1}{2}\partial_{xx}(\sigma^2 p).$$

When deriving moment equations or stationary densities, state the boundary conditions needed to discard boundary terms. You may use Itô's formula and standard facts about Gaussian and lognormal distributions.

Problem 1. Let

$$dX_t = -2X_t\,dt + dW_t, \qquad X_0 = x_0.$$

(a) Derive the Fokker–Planck equation. Solve the SDE and hence obtain the mean, variance, and density of X_t.

(b) For $x_0 = 0$, verify directly that

$$p(t, x) = \frac{1}{\sqrt{2\pi v(t)}} \exp\left(-\frac{x^2}{2v(t)}\right), \qquad v(t) = \frac{1 - e^{-4t}}{4},$$

satisfies that equation.

(c) Find the limiting density as $t \to \infty$ and verify that it is a stationary solution of the Fokker–Planck equation.

(d) For $x_0 = 0$, plot $p(t, x)$ for $t = 0.1, 0.5, 1$, together with the stationary density, on the same axes.

Problem 2. Let

$$dX_t = 2\,dt + dW_t, \qquad X_0 = 0.$$

(a) Determine the density of X_t and verify it in the corresponding Fokker–Planck equation.

(b) Starting only from that equation, derive and solve differential equations for $m_1(t) = \mathbb{E}[X_t]$ and $m_2(t) = \mathbb{E}[X_t^2]$. Deduce $\mathrm{Var}(X_t)$.

(c) Prove that the process has no stationary probability density on $\mathbb{R}$.

Problem 3. Suppose

$$dX_t = X_t\,dt + X_t\,dW_t, \qquad X_0 = 1.$$

(a) Apply Itô's formula to $\log X_t$ and determine the law of X_t.

(b) Derive the Fokker–Planck equation on $(0, \infty)$ and verify that

$$p(t,x) = \frac{1}{x\sqrt{2\pi t}} \exp\left(-\frac{(\log x - t/2)^2}{2t}\right), \qquad x > 0,$$

is a solution.

(c) Compute the mean, variance, and mode of X_t.

(d) The mean of X_t increases while its mode decreases. Explain this apparent contrast by using the formulas from the preceding part to identify which feature of the lognormal density pulls the mean to the right.

Problem 4. For a time-homogeneous diffusion $dX_t = b(X_t)\,dt + \sigma\,dW_t$ on $\mathbb{R}$, define the probability current

$$J(x) = b(x)p(x) - \frac{\sigma^2}{2}p'(x).$$

Assume that a stationary density has zero current at infinity.

(a) Show that stationarity implies J is constant and hence $J \equiv 0$. Deduce a first-order differential equation for p.

(b) Apply this equation to

$$dX_t = -aX_t\,dt + \sqrt{2}\,dW_t, \qquad a > 0,$$

and find the normalized stationary density and its variance.

(c) Apply it to

$$dX_t = -\frac{X_t}{1 + X_t^2}\,dt + dW_t$$

and find the normalized stationary density. Determine which of its first and second absolute moments are finite.

(d) Compare the two stationary tail probabilities by finding the leading asymptotic order of $\mathbb{P}(|X| > r)$ as $r \to \infty$ in each case.