Stochastic Processes 2026–27

Problem Set 11: Itô Calculus Practice

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Instructions. Throughout, $(B_t)_{t\geq 0}$ is a standard Brownian motion. You may use Itô's formula and the standard moments of the normal distribution. State clearly any other theorem you use.

Problem 1. For each process below, find adapted processes μ_t and σ_t such that

$$dY_t = \mu_t\, dt + \sigma_t\, dB_t.$$

(a) $Y_t = \sin(B_t)$;

(b) $Y_t = (B_t)^p$, where $p > 0$;

(c) $Y_t = \exp(B_t - t^2)$;

(d) $Y_t = \log(B_t)$;

(e) $Y_t = t^2 B_t$.

For parts (b) and (d), first state the values of p and the stochastic intervals, respectively, on which the expressions are real-valued and Itô's formula applies in its standard form.

Problem 2. Define

$$X_t = \int_0^{B_t} e^{-s^2}\, ds, \qquad Y_t = \exp\left(\int_0^t B_s\, ds\right), \qquad Z_t = \frac{t}{X_t}.$$

Compute the quadratic variation of each process. For Z, first determine where the expression defines a process, and give its quadratic variation on any stochastic interval on which X does not vanish.

Problem 3. Define

$$X_t = \int_0^t s\, dB_s, \qquad Y_t = \int_0^t B_s\, ds.$$

(a) Find formulas for $\mathbb{E}[X_t^n]$ and $\mathbb{E}[Y_t^n]$ for every $n \in \mathbb{N}$.

(b) Do X and Y have identical finite-dimensional distributions? Justify your answer.

Problem 4. Determine whether the identity

$$e^{2t}\sin(2B_t) = 2\int_0^t e^{2s}\cos(2B_s)\,dB_s$$

is true or false, and justify your answer.

Problem 5. Consider the system

$$dX_t = -Y_t\,dB_t - \frac{1}{2}X_t\,dt, \qquad dY_t = X_t\,dB_t - \frac{1}{2}Y_t\,dt.$$

Suppose $(X_0, Y_0) = (x, y)$, where $x^2 + y^2 = 1$. Show that $X_t^2 + Y_t^2 = 1$ for every $t \geq 0$, and hence that the diffusion remains on the unit circle. Give an intuitive explanation for this invariance.

Problem 6. Consider the SDE

$$dX_t = tX_t\,dt + e^{t^2/2}\,dB_t, \qquad X_0 = 1.$$

(a) Solve the SDE explicitly.

(b) Compute $\mathbb{E}[X_t]$ and $\mathrm{Var}(X_t)$.

(c) Write the Fokker–Planck equation for the density $p(t, x)$ of X_t.

(d) Use the integrating factor from part (a) to introduce a new spatial variable and a correspondingly rescaled density. Show that, in the new variables, the Fokker–Planck equation reduces to the heat equation.