Stochastic Processes 2026–27

Problem Set 8: Absorbing Chains and Word Problems

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Instructions. You may use the axioms of a probability measure, the Chapman–Kolmogorov identity, and the results proved in class. Unless a part asks for an exact expression, give numerical answers to at least four significant figures, and state the order of the states whenever you display a transition matrix. You may quote the results on absorbing chains proved in class (the canonical form, the fundamental matrix, and its consequences), *except in Problem 1*, which asks you to build that theory from scratch. Choosing which tool a question calls for is part of the exercise.

Problem 1. (Absorption is certain: $Q^n \to 0$.) Let P be a finite absorbing chain in canonical form with a nonempty set $\mathcal{T}$ of transient states. *Do not assume $I - Q$ is invertible, and do not use the fundamental matrix.* Prove that $Q^n \to 0$ entrywise, through the steps below.

(a) Show that from any transient state i the expected total number of visits to a transient state j is finite,

$$\mathbb{E}_i\Big[\textstyle\sum_{n\ge 0} \mathbf{1}\{X_n = j\}\Big] = \sum_{n\ge 0}(Q^n)_{ij} = \sum_{n\ge 0} p_{ij}^{(n)} < \infty.$$

(Reach j once, then a geometric number of returns; j transient gives $f_{jj} < 1$.)

(b) Deduce that, started from i, each transient j is visited only finitely often almost surely, so $\Pr_i(X_n = j \text{ for infinitely many } n) = 0$. By a union bound over the finitely many transient states, conclude

$$\Pr_i\big(\text{some transient state is visited infinitely often}\big) = 0.$$

(c) Let $A_n = \{X_n \in \mathcal{T}\}$. Show the events A_n are decreasing, and that on $\bigcap_n A_n$ (the chain never leaves $\mathcal{T}$) some transient state must be visited infinitely often. Conclude $\Pr_i\big(\bigcap_n A_n\big) = 0$, and hence $\Pr_i(A_n) \to 0$.

(d) Show $\Pr_i(A_n) = \sum_{j\in\mathcal{T}}(Q^n)_{ij}$, and deduce from $0 \le (Q^n)_{ij} \le \Pr_i(A_n)$ that $Q^n \to 0$.

Problem 2. (The fixed die.) A die is altered so that it never shows the same face twice in a row: if the previous roll was i, the next roll is equally likely to be any of the other five faces. Let X_n be the nth roll, and suppose $X_1 = 6$.

(a) Write the transition matrix on $\{1, 2, 3, 4, 5, 6\}$.

(b) Let $a_n = \Pr(X_{n+1} = 6 \mid X_1 = 6)$. Find a recursion for a_n with its initial condition, and solve it for a closed form.

(c) Find $\Pr(X_{n+1} = 1 \mid X_1 = 6)$, and state its limit as $n \to \infty$.

Problem 3. (Ehrenfest urn.) Four balls are distributed between urns A and B. At each step one of the four balls is chosen uniformly at random and moved to the other urn. Let X_n be the number of balls in urn A.

(a) Write the state space and the transition matrix, in increasing order of the states.

(b) Find a stationary distribution π, and verify the detailed-balance equations $\pi_i P_{ij} = \pi_j P_{ji}$.

(c) Find the mean return times to state 0 and to state 2, using the identity $\nu_i = 1/\pi_i$ proved in class.

Problem 4. (Inventory with random demand.) A shop stocks one item; the shelf holds at most 3 units. Let $X_n \in \{0, 1, 2, 3\}$ be the units left at the end of day n. Overnight: if $X_n \in \{0, 1\}$ the shelf is refilled to 3; if $X_n \in \{2, 3\}$ no order is placed. Daily demand D is independent across days with

$$\Pr(D = 0) = 0.1, \quad \Pr(D = 1) = 0.4, \quad \Pr(D = 2) = 0.3, \quad \Pr(D = 3) = 0.2,$$

and demand above the available stock is lost.

(a) Write the transition matrix of (X_n), in the order $(0, 1, 2, 3)$.

(b) Find the stationary distribution.

(c) Find the long-run fraction of nights on which an order is placed, and the long-run expected number of units ordered per night.

(d) Find the long-run expected end-of-day inventory and the long-run expected number of lost sales per day.

Problem 5. (Gambler's ruin.) A gambler plays a fair game repeatedly. Each round he stakes Rs. 10,000: with probability $\frac{1}{2}$ he gains Rs. 10,000 and with probability $\frac{1}{2}$ he loses it. He starts with Rs. 20,000 and stops the moment he is bankrupt (Rs. 0) or reaches his target of Rs. 40,000. Let X_n be his capital after n rounds in units of Rs. 10,000, so $X_n \in \{0, 1, 2, 3, 4\}$ with 0 and 4 absorbing and $X_0 = 2$.

(a) Write the transition matrix and identify the absorbing states.

(b) Starting from Rs. 20,000, find the expected number of times the gambler holds each of the levels Rs. 10,000, Rs. 20,000, Rs. 30,000 before the game ends.

(c) Find the expected number of rounds he plays before the game ends.

(d) Find the probability that he reaches Rs. 40,000 and the probability that he goes bankrupt.

Problem 6. (The bold gambler.) A gambler has £2 and wants £10. Each play a fair coin is tossed; a win pays an amount equal to the stake, a loss forfeits it. He plays boldly: if his capital is £5 or less he stakes all of it; if it is more than £5 he stakes just enough that a win takes him to £10. Let X_n be his capital; the game ends at 0 or 10.

(a) Starting from $X_0 = 2$, list the reachable states and write the transition matrix, stating the order and marking the absorbing states.

(b) Find the probability that, starting from £2, he reaches £10 before going broke.

(c) Find the expected number of tosses until the game ends, starting from £2.

Problem 7. (A monkey types a pattern.) A monkey presses the keys `A` and `B` independently, each with probability $\frac{1}{2}$. It stops the instant it has just typed the block

$$\texttt{A A B A B}.$$

Track its progress by the length of the longest suffix of what has been typed so far that is a prefix of `AABAB`. This gives states $0, 1, 2, 3, 4, 5$, with 5 absorbing.

(a) A mismatch need *not* send progress back to 0: for instance, from progress 4 (having matched `AABA`) a further `A` leaves the suffix `AA`, which is progress 2. Give the transition out of each state on reading `A` and on reading `B`.

(b) Find the expected number of keystrokes to first type `AABAB`.

(c) Compare this with the answer the (incorrect) rule "any wrong key resets progress to 0" would give. Which is larger, and why?

Remark. This is the mechanism behind "monkeys typing Shakespeare": for a target of length L over an a-key board the same automaton gives an expected wait of order a^L, astronomically large for a play but finite.

Problem 8. (Credit ratings and a defaultable bond.) A bond is rated good (G), risky (R), or in default (D); default is absorbing. Each year the rating changes by

$$P = \begin{pmatrix} 0.88 & 0.10 & 0.02 \\ 0.08 & 0.82 & 0.10 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{order } (G, R, D).$$

(a) Identify the transient and absorbing states. Starting from G, and from R, find the probability of default within two years.

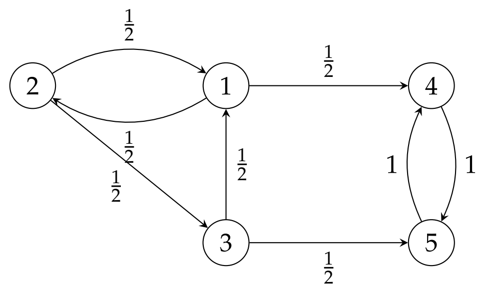
(b) Find the expected time to default, starting from G and from R.

(c) A two-year zero-coupon bond pays Rs. 100 at the end of year 2 if it has not defaulted by then and Rs. 0 otherwise. With an annual risk-free rate of 5%, compute its present value if currently rated G.

(d) Suppose instead default pays a recovery of Rs. 40 at the end of the year in which default occurs. Recompute the present value starting from G.

Problem 9. (Time to reach a closed class.) A DTMC on $S = \{1,2,3,4,5\}$ has the transition matrix and diagram below: the states $\{1,2,3\}$ form one communicating class and $\{4,5\}$ another.

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \text{rows and columns ordered } 1,2,3,4,5.$$



(a) Find the two communicating classes. State which is open and which is closed, and classify each (transient, null recurrent, or positive recurrent), justifying each verdict by a theorem.

(b) Find the expected number of steps until the chain first enters the closed class, starting from each of the states 1, 2, and 3.