Stochastic Processes 2026–27

Problem Set 7

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Instructions. You may use the axioms of a probability measure, the Chapman–Kolmogorov identity, and the results proved in class. No other results may be assumed without proof.

Problem 1. A bank reviews each home-loan borrower once a month and records one of five statuses: current (C), delinquent (D), restructured (S), defaulted (X), or paid off (P). Default and pay-off are permanent. From *current* the borrower stays current with probability 0.8, slips to delinquent with probability 0.1, and pays the loan off with probability 0.1. From *delinquent* the borrower catches up to current with probability 0.3, stays delinquent with probability 0.2, is restructured with probability 0.3, and defaults with probability 0.2. From *restructured* the borrower becomes current with probability 0.4, slips to delinquent with probability 0.2, stays restructured with probability 0.3, and defaults with probability 0.1.

(a) Write the transition matrix, ordering the states C, D, S, X, P.

(b) Find the communicating classes, and state which are closed.

(c) Classify each state as recurrent or transient, justifying each verdict by a theorem.

(d) Show that, whatever the starting status, the borrower is absorbed into default or pay-off with probability one, and that each of C, D, S is occupied only finitely often.

Problem 2. A DTMC on $S = \{1, 2, 3, 4, 5, 6\}$ has transition matrix, with rows and columns ordered $1, \dots, 6$,

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Draw the transition diagram.

(b) Find the communicating classes, and state which are closed.

(c) Classify each state as recurrent or transient, with justification.

Problem 3. A DTMC on $S = \{0, 1, 2, \dots\}$ moves from state i to state $i+1$ with probability p_i and to state 0 with probability $1 - p_i$, where $0 < p_i < 1$ for every i.

(a) Show that the chain is irreducible.

(b) Determine, in terms of the sequence $(p_i)_{i\geq 0}$, whether the chain is recurrent or transient.

(c) Classify the chain in each of the cases

$$p_i = \frac{i+1}{i+2} \qquad \text{and} \qquad p_i = 1 - 2^{-(i+1)}.$$

You may use that, for $a_i \in (0, 1)$,

$$\prod_i (1 - a_i) > 0 \qquad \Longleftrightarrow \qquad \sum_i a_i < \infty.$$

Problem 4. In class we showed that the simple symmetric random walk on $\mathbb{Z}$ is recurrent. Prove that it is null recurrent.

Problem 5. Let $M_j = \#\{\, n \geq 1 : X_n = j \,\}$ be the number of visits to j.

(a) Suppose

$$\sum_{n\geq 1} p_{jj}^{(n)} < \infty.$$

Using the first Borel–Cantelli lemma with the events $A_n = \{X_n = j\}$ under $\Pr(\,\cdot \mid X_0 = j)$, show that

$$\Pr\big(M_j < \infty \mid X_0 = j\big) = 1.$$

(b) Suppose instead j is recurrent ($f_{jj} = 1$), and use that

$$\Pr\big(M_j \geq k \mid X_0 = j\big) = f_{jj}^{\,k}, \qquad k \geq 0.$$

Show that $\Pr(M_j = \infty \mid X_0 = j) = 1$, so that, started from j, the chain returns to j at infinitely many finite times almost surely.

Problem 6. Let the state space S be finite.

(a) Prove that the chain has at least one closed communicating class.

(b) Deduce that at least one state is recurrent.

(c) Conclude that if the chain is irreducible, then every state is recurrent.

Optional: generating functions and the mean return time

This section re-proves the null recurrence of Problem 4 by an exact method, and on the way recovers the recurrence shown in class.

For a sequence $(a_n)_{n\geq 0}$ of nonnegative reals, its *generating function* is

$$A(s) = \sum_{n=0}^{\infty} a_n\, s^n, \qquad s \in [0,1),$$

the discrete analogue of the Laplace transform $\int_0^\infty e^{-\lambda t} a(t)\, dt$, with $s = e^{-\lambda}$ in place of the decay factor. Three facts, used freely below.

- **Addition.** If $c_n = a_n + b_n$, then
$$C(s) = A(s) + B(s).$$
- **Convolution.** If $c_n = \sum_{k=0}^{n} a_k\, b_{n-k}$, then
$$C(s) = A(s)\, B(s).$$
- **Mean.** $A'(s) = \sum_{n\geq 1} n\, a_n\, s^{n-1}$, and for nonnegative terms
$$\sum_{n\geq 1} n\, a_n = \lim_{s\to 1^-} A'(s) \in [0,\infty].$$

For the simple symmetric random walk on $\mathbb{Z}$ started at 0, write

$$P(s) = \sum_{n\geq 0} p_{00}^{(n)}\, s^n \quad (p_{00}^{(0)} = 1), \qquad F(s) = \sum_{n\geq 1} f_{00}^{(n)}\, s^n.$$

Problem 7. (a) Show that

$$p_{00}^{(2n+1)} = 0 \qquad \text{and} \qquad p_{00}^{(2n)} = \binom{2n}{n} 4^{-n}.$$

(b) Using the generalized binomial theorem, show that

$$\sum_{n\geq 0} \binom{2n}{n} x^n = (1-4x)^{-1/2}, \qquad |x| < \tfrac{1}{4}.$$

(c) Deduce that $P(s) = (1-s^2)^{-1/2}$.

Problem 8. *An alternative to Problem 7, via characteristic functions.* Let $\xi = \pm 1$ with probability $\frac{1}{2}$ be a single step, so that $X_n = \xi_1 + \cdots + \xi_n$.

(a) Show that $\varphi(t) = \mathbb{E}[e^{it\xi}] = \cos t$, and hence

$$\mathbb{E}[e^{itX_n}] = \cos^n t.$$

(b) Using

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(kt)\, dt = \mathbf{1}\{k=0\},$$

deduce that

$$p_{00}^{(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^n t\, dt.$$

(c) Sum the geometric series and evaluate the resulting integral by the substitution $u = \tan(t/2)$ to recover $P(s) = (1-s^2)^{-1/2}$.

Remark. The same method on $\mathbb{Z}^d$ gives

$$\sum_n p_{00}^{(n)} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{dt}{1 - d^{-1}\sum_j \cos t_j},$$

whose singularity at $t = 0$ is integrable exactly when $d \geq 3$: the walk is recurrent for $d \leq 2$ and transient for $d \geq 3$ (Pólya).

Problem 9. Prove that, for $n \geq 1$,

$$p_{00}^{(n)} = \sum_{k=1}^{n} f_{00}^{(k)}\, p_{00}^{(n-k)}.$$

Problem 10. Using the identity of Problem 9 and the convolution rule, deduce that

$$P(s) = 1 + F(s)\,P(s), \qquad \text{hence} \qquad F(s) = 1 - \frac{1}{P(s)}.$$

With the closed form of P, obtain $F(s) = 1 - \sqrt{1-s^2}$.

Problem 11. (a) Show that $f_{00} = F(1) = 1$, so the walk is recurrent.

(b) Compute $F'(s)$ and show that

$$\lim_{s\to 1^-} F'(s) = \infty.$$

Conclude that the mean return time

$$\nu_0 = \sum_{n \geq 1} n \, f_{00}^{(n)}$$

is infinite, so the walk is null recurrent.