Stochastic Processes 2026–27

Problem Set 6

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21 August 2026

Instructions. You may use the axioms of a probability measure and the definitions below. No other results may be assumed without proof. No hints are given.

Recall. Throughout, $(\Omega, \mathcal{F}, \Pr)$ is a probability space.

- A *discrete-time Markov chain* (DTMC) on a countable state space S is a process $\{X_n\}_{n\geq 0}$ with, for all states and all n (whenever the conditioning event has positive probability),

  $$\Pr(X_{n+1} = y \mid X_0 = x_0, \dots, X_{n-1} = x_{n-1}, X_n = x) = \Pr(X_{n+1} = y \mid X_n = x) =: P_{xy},$$

  with P_{xy} not depending on n. The matrix $P = (P_{xy})$ is the *transition matrix*; $P_{xy} \geq 0$ and $\sum_y P_{xy} = 1$.

- For states i, j and $n \geq 1$, the *first passage probability*

  $$f_{ij}^{(n)} := \Pr(X_1 \neq j, \dots, X_{n-1} \neq j, X_n = j \mid X_0 = i),$$

  and the *eventual passage probability* $f_{ij} := \Pr(\exists\, n \geq 1 : X_n = j \mid X_0 = i) = \sum_{n\geq 1} f_{ij}^{(n)}$.

Problem 1 (A stock-market watcher). *A share trades at an integer price $X_n \in \{5, 6, 7, 8, 9, 10\}$, in thousands of rupees (kRs), in quarter n. Each quarter the price rises by one, falls by one, or holds, according to a time-homogeneous DTMC: from price i it moves to $i+1$ with probability p_i, to $i-1$ with probability q_i, and stays at i with probability $r_i = 1 - p_i - q_i$, where*

i	5	6	7	8	9	10
p_i *(up)*	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0
q_i *(down)*	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
r_i *(hold)*	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

(At price 5 the share cannot fall further; at price 10 it cannot rise.)

A trader enters in quarter 0 with the price at $X_0 = 8$, holding no shares. His strategy is: buy one dozen (12) shares the first quarter the price goes below 7 while he holds none, and sell the dozen the first quarter the price goes above 9 while he holds a dozen.

(a) *Write down the transition matrix P, with the states ordered* $5, 6, 7, 8, 9, 10$.

(b) *Find* $\mathbb{E}[X_3 \mid X_2, X_1]$.

(c) *What is the probability that the trader never buys the shares?*

(d) *What is the probability that he buys in the* 4*th quarter?*

(e) *Compute the trader's expected expenditure in the* 3*rd quarter.*

(f) *Show that once he has bought the dozen he sells it with probability* 1*, and hence find his profit, in kRs, on a completed buy–then–sell round.*

(g) *Using the strong Markov property, find the probability that he sells the dozen in the* 4*th quarter after the quarter he bought it.*

Problem 2 (Two urns)**.** *Two urns A and B each contain* 10 *balls. Initially A holds* 10 *red balls and B holds* 10 *white balls. At each step one ball is drawn uniformly at random from each urn, and the two drawn balls are swapped, each moved to the other urn. The step is repeated forever.*

(a) *Describe a DTMC for the state of the system, and give its state space.*

(b) *Write the transition matrix P.*

(c) *Write a Python program to compute P^{1000} and P^{1001}.*

(d) *Interpret the results.*

Problem 3 (A gambler with a goal)**.** *A gambler starts with* $10k$ *rupees and bets* $1k$ *per game, winning each game with probability p and losing the stake otherwise, independently across games. He plays until he reaches his goal of* $15k$ *rupees or goes broke at* 0.

(a) *Model the gambler's fortune as a DTMC, and give its state space.*

(b) *Write the transition matrix P.*

(c) *What is the probability of reaching* $13k$ *rupees in exactly five steps?*

Problem 4 (Runs of heads)**.** *Let $(X_n)_{n\geq 1}$ be independent tosses of a biased coin, showing heads* (1) *with probability p and tails* (0) *otherwise. Let R_n be the current run of heads at time n: the number of consecutive heads ending at time n, and* 0 *if $X_n = 0$. For example, the toss sequence* 1100010011100 *gives the run sequence*

$$R = [1, 2, 0, 0, 0, 1, 0, 0, 1, 2, 3, 0, 0].$$

Prove that $(R_n)_{n\geq 1}$ is a DTMC, and give its transition probabilities.

Problem 5 (Inventory with reordering). *A store stocks a product up to a maximum of $S \in \mathbb{N}$ units. Each day's demand $D_t \sim \mathrm{Uniform}\{0, 1, \ldots, S\}$ ($S + 1$ equally likely values) is i.i.d. across days, and demand beyond the stock on hand is lost. The store reviews the stock at the end of each day: if the shelf is empty it reorders and is refilled to S by the next morning, and otherwise it does not reorder. Let the state be the inventory on hand at the start of the day, after any restocking.*

(a) *Prove that the inventory level is a DTMC.*

(b) *Write the transition matrix for $S = 3$.*

Problem 6 (Loan delinquency). *A borrower's status at time t is the number of months behind on loan payments, ranging over $\{0, 1, \ldots, B\}$; at B the borrower is declared to be in default and remains there permanently. At each step, independently of the past:*

- *with probability p the borrower makes a payment, which reduces the delay by one month if behind, and changes nothing if fully caught up (status 0);*
- *with probability $1 - p$ the borrower misses the payment, which increases the delay by one month, except at the default limit B, where the status is unchanged.*

Construct a DTMC for the borrower's status: give the state space and the transition probabilities.