Stochastic Processes 2026–27

Problem Set 5

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Instructions. You may use the axioms of a probability measure, the properties of conditional expectation (G1, G2, tower, linearity, taking out what is known), and the definitions below. No other results may be assumed without proof. No hints are given.

Recall. Throughout, $(\Omega, \mathcal{F}, \Pr)$ is a probability space.

- A *filtration* is an increasing family $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$ of sub-σ-algebras. A process $\{X_n\}_{n\geq 0}$ is *adapted* to $\{\mathcal{F}_n\}$ if each X_n is $\mathcal{F}_n$-measurable.
- An adapted process $\{X_n\}$ with $\mathbb{E}|X_n| < \infty$ for all n is a *martingale* with respect to $\{\mathcal{F}_n\}$ if
$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n \qquad \text{for every } n \geq 0.$$
- A *discrete-time Markov chain* (DTMC) on a countable state space S is a process $\{X_n\}_{n\geq 0}$ with, for all states and all n (whenever the conditioning event has positive probability),
$$\Pr(X_{n+1} = y \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x) = \Pr(X_{n+1} = y \mid X_n = x) =: P_{xy},$$
with the value P_{xy} not depending on n (time homogeneity). The matrix $P = (P_{xy})$ is the *transition matrix*; $P_{xy} \geq 0$ and $\sum_y P_{xy} = 1$ for each x.
- A function $h : S \to \mathbb{R}$ is *harmonic* for P if
$$h(x) = \sum_{y\in S} P_{xy}\, h(y) \qquad \text{for every } x \in S.$$
- A probability vector $\pi = (\pi_x)_{x\in S}$ is an *invariant* (stationary) distribution if $\pi_y = \sum_x \pi_x P_{xy}$ for every y, that is $\pi P = \pi$.

Problem 1 (Martingales look forward, not just one step)**.** *Let $\{X_n\}_{n\geq 0}$ be a martingale with respect to a filtration $\{\mathcal{F}_n\}_{n\geq 0}$. Prove that for all integers $m > n \geq 0$,*
$$\mathbb{E}[X_m \mid \mathcal{F}_n] = X_n.$$

Problem 2 (Every random variable generates a martingale). *Let X be an integrable random variable on $(\Omega, \mathcal{F}, \Pr)$, and let $\{\mathcal{F}_n\}_{n\geq 0}$ be a filtration. Define*

$$X_n := \mathbb{E}[X \mid \mathcal{F}_n], \qquad n \geq 0.$$

Show that $\{X_n\}_{n\geq 0}$ is a martingale with respect to $\{\mathcal{F}_n\}$.

Problem 3 (Harmonic functions along a Markov chain). *Let $\{X_n\}_{n\geq 0}$ be a DTMC on a countable state space S with transition matrix P, and let $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$. Let $f : S \to \mathbb{R}$ be a harmonic function for P, and assume $\mathbb{E}|f(X_n)| < \infty$ for every n (automatic when f is bounded or S is finite). Define*

$$M_n := f(X_n), \qquad n \geq 0.$$

Show that $\{M_n\}_{n\geq 0}$ is a martingale with respect to $\{\mathcal{F}_n\}$.

Problem 4 (The lily-pad frog). *A frog lives on two lily pads, east e and west w. Each pad carries its own coin. Every morning the frog tosses the coin on the pad it currently occupies: on heads it jumps to the other pad, on tails it stays. The pad's coin on e shows heads with probability p and the pad's coin on w shows heads with probability q, where $p, q \in (0, 1]$. The sequence of occupied pads $X_0, X_1, X_2, \dots \in \{e, w\}$ is a DTMC.*

(a) *Write down the transition matrix P.*

(b) *Find* all *harmonic functions $f : \{e, w\} \to \mathbb{R}$ for P.*

(c) *State what your answer to (b) says about the martingale $M_n = f(X_n)$.*

Problem 5 (Coupon collector as a Markov chain). *There are m coupon types. At each draw one of the m types is chosen, independently across draws and uniformly at random. Let*

$$X_n = \textit{number of distinct coupon types collected after } n \textit{ draws}, \qquad X_0 = 0.$$

(a) *Show that $\{X_n\}_{n\geq 0}$ is a DTMC on $\{0, 1, \dots, m\}$.*

(b) *Find its transition probabilities P_{jk} for $j, k \in \{0, \dots, m\}$.*

Problem 6 (Ehrenfest urns). *N balls are distributed between two urns A and B. At each step one of the N balls is selected uniformly at random and transferred to the other urn. Let X_n be the number of balls in urn A after n steps.*

(a) *Show that $\{X_n\}_{n\geq 0}$ is a DTMC on $\{0, 1, \dots, N\}$, and give its transition probabilities for general N.*

(b) *Write out the transition matrix explicitly for $N = 3$.*

(c) *Find the invariant distribution π on the states for $N = 3$.*