Stochastic Processes 2026–27

Problem Set 4

Srikanth Pai

Assistant Professor, Madras School of Economics

8 August 2026

Instructions. You may use the definitions and results of Chapter 3 (stochastic processes), Sections 1–4, together with the definitions of discrete- and continuous-time processes, chains, and the Markov property. No other results may be assumed without proof. No hints are given.

Recall. Throughout, $(\Omega, \mathcal{F}, \Pr)$ is a probability space.

- A *stochastic process* is a family $\{X_t\}_{t\in I}$ of random variables on $(\Omega, \mathcal{F}, \Pr)$; for a fixed outcome ω, the map $t \mapsto X_t(\omega)$ is a *sample path*.
- The *finite-dimensional distributions* (FDDs) are the joint laws of $(X_{t_1}, \ldots, X_{t_k})$ over all finite tuples of distinct indices, recorded by the joint CDFs

$$F_{t_1,\ldots,t_k}(x_1, \ldots, x_k) = \Pr(X_{t_1} \le x_1, \ldots, X_{t_k} \le x_k);$$

  for an integer-valued process it is equivalent to give the joint pmfs.
- A family of candidate FDDs is *consistent* if **(K1, symmetry)** permuting the indices and their arguments together leaves the value unchanged, and **(K2, marginalisation)** sending one argument to $+\infty$, equivalently summing a joint pmf over one coordinate, returns the FDD of the remaining indices. By Kolmogorov's theorem a family is the FDD family of some process if and only if it is consistent.
- $\{X_t\}$ is *strictly stationary* if its FDDs are invariant under a common time shift: $F_{t_1+j,\ldots,t_k+j} = F_{t_1,\ldots,t_k}$ for all indices $t_1, \ldots, t_k$ and every shift j.
- $\{X_t\}$ with $\mathbb{E}[X_t^2] < \infty$ is *covariance-stationary* if $\mathbb{E}[X_t]$ does not depend on t and $\mathrm{Cov}(X_t, X_{t+k})$ depends only on the lag k.
- A process is *discrete-time* if I is countable and *continuous-time* if I is an interval of $\mathbb{R}$; it is a *chain* if the set of values each X_t can take is at most countable.
- A discrete-time process has the *Markov property* if $\mathbb{E}[g(X_n) \mid X_1, \ldots, X_{n-1}] = \mathbb{E}[g(X_n) \mid X_{n-1}]$ for every bounded g; for a chain this reads

$$\Pr(X_n = x_n \mid X_1 = x_1, \ldots, X_{n-1} = x_{n-1}) = \Pr(X_n = x_n \mid X_{n-1} = x_{n-1})$$

  whenever the conditioning event has positive probability.

Problem 1 (Which families are finite-dimensional distributions?). *For each family below, decide whether it is the family of finite-dimensional distributions of some stochastic process, proving consistency or exhibiting the condition that fails.*

(a) *Index set $I = \mathbb{Z}$; fix $c \in \mathbb{R}$ and set*

$$F_{t_1,\dots,t_k}(x_1,\dots,x_k) = \prod_{i=1}^{k} \mathbf{1}\{c \le x_i\}.$$

Is the family consistent? Is the resulting process strictly stationary?

(b) *Index set $I = \mathbb{N}$; for distinct $n_1,\dots,n_k$ and $a_1,\dots,a_k \in \{0,1\}$,*

$$\Pr(X_{n_1} = a_1,\dots,X_{n_k} = a_k) = 2^{-k}.$$

Is the family consistent?

(c) *Index set $I = \mathbb{N}$; the single-time law is $\Pr(X_n = 1) = \Pr(X_n = 0) = \frac{1}{2}$, and for each pair $m \ne n$ the joint law is*

$$\begin{aligned}
&\Pr(X_m = 1, X_n = 1) = \tfrac{1}{2}, \quad \Pr(X_m = 0, X_n = 0) = \tfrac{1}{4},\\
&\Pr(X_m = 1, X_n = 0) = \tfrac{1}{4}, \quad \Pr(X_m = 0, X_n = 1) = 0.
\end{aligned}$$

Can this be extended to the FDD family of a process?

Problem 2 (A parity process). *A fair coin is tossed repeatedly and independently. Let H_n be the number of heads in the first n tosses, and define*

$$X_n = \begin{cases} 1, & H_n \text{ is odd,} \\ 0, & H_n \text{ is even,} \end{cases} \qquad n = 0,1,2,\dots,$$

so that $X_0 = 0$.

(a) *Is $\{X_n\}$ a discrete-time or a continuous-time process? Is it a chain?*

(b) *Determine whether $\{X_n\}$ has the Markov property.*

(c) *With the process started from $X_0 = 0$ as above, determine whether it is strictly stationary.*

(d) *(Optional challenge.) Work out the finite-dimensional distributions: for $0 \le n_1 < n_2 < \cdots < n_k$, give the joint pmf of $(X_{n_1},\dots,X_{n_k})$.*

Problem 3 (Stationary or not). (a) *Show that every IID process $\{X_t\}_{t\in\mathbb{Z}}$ is strictly stationary.*

(b) *Fix a single random variable Z and set $X_t = Z$ for all $t \in \mathbb{Z}$. Show that $\{X_t\}$ is strictly stationary.*

(c) *Show that a strictly stationary process with $\mathbb{E}[X_t^2] < \infty$ is covariance-stationary. Is the converse true? Justify.*