Stochastic Processes 2026–27

Problem Set 3

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Instructions. You may use the axioms of a σ-algebra and of a probability measure, results proved in L01–L04, and standard rules of set theory. No other results may be assumed without proof. Venn diagrams are not proofs.

Recall (from L04). Throughout, $(\Omega, \mathcal{F}, \Pr)$ is a probability space and all random variables are $\mathcal{F}$-measurable unless stated otherwise.

- $X : \Omega \to \mathbb{R}$ is *measurable with respect to* $\mathcal{F}$ if $X^{-1}(B) \in \mathcal{F}$ for every Borel set B (equivalently $\{X \leq b\} \in \mathcal{F}$ for every $b \in \mathbb{R}$).
- The σ-algebra *generated by* X is $\sigma(X) := \sigma\big(\{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}\big)$, the smallest σ-algebra making X measurable.
- An *atom* of $\mathcal{F}$ is a non-empty $A \in \mathcal{F}$ whose only measurable proper subset is $\emptyset$.
- X is *integrable* if $\mathbb{E}[|X|] < \infty$; the expectation of an indicator is $\mathbb{E}[\mathbf{1}_A] = \Pr(A)$.
- For a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, a random variable Z is the *conditional expectation* $\mathbb{E}[X \mid \mathcal{G}]$ if **(G1)** Z is $\mathcal{G}$-measurable, and **(G2)** $\mathbb{E}[X\,\mathbf{1}_G] = \mathbb{E}[Z\,\mathbf{1}_G]$ for every $G \in \mathcal{G}$.
- *Finite-partition formula.* If $C_1, \ldots, C_r$ partition Ω into atoms of $\mathcal{G} = \sigma(\{C_1, \ldots, C_r\})$ with $\Pr(C_i) > 0$, then
$$\mathbb{E}[X \mid \mathcal{G}] = \sum_{i=1}^{r} \frac{\mathbb{E}[X\,\mathbf{1}_{C_i}]}{\Pr(C_i)}\,\mathbf{1}_{C_i}.$$
- *Doob–Dynkin theorem.* Y is $\sigma(X)$-measurable if and only if $Y = f(X)$ for some Borel measurable $f : \mathbb{R} \to \mathbb{R}$ (that is, $f^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for every Borel B).
- *Uniform probability space on* $[0,1]$*:* $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, $\Pr([a,b]) = b - a$, and $\mathbb{E}[Z] = \int_0^1 Z(u)\,\mathrm{d}u$.

Problem 1. *Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra.*

(a) *Prove that every $\mathcal{G}$-measurable random variable is $\mathcal{F}$-measurable.*

(b) *A fair coin is tossed three times, Ω is the eight outcomes, $\mathcal{F} = 2^{\Omega}$, and $\mathcal{G} = \sigma(C)$ where $C = \{$first toss is $H\}$. Give an explicit $\mathcal{F}$-measurable random variable that is* not *$\mathcal{G}$-measurable, and prove it is not.*

Problem 2. *Four fair coins are tossed, so* Ω *is the set of the* 16 *length-four strings over* $\{H, T\}$*, with* $\mathcal{F} = 2^{\Omega}$*. Let* $X(\omega)$ *be the number of times a head is immediately followed by a tail in* ω*; for instance* $X(HTHT) = 2$*,* $X(HHTT) = 1$*, and* $X(HHHH) = 0$*. Determine* $\sigma(X)$*: list its atoms explicitly, and state how many sets it contains.*

Problem 3. *Let* $X : \Omega \to \mathbb{R}$ *and let* $f : \mathbb{R} \to \mathbb{R}$ *be Borel measurable, and set* $Y = f(X)$*.*

(a) *Prove directly from the definition that* Y *is* $\sigma(X)$*-measurable. (This is the easy half of the Doob–Dynkin theorem.)*

(b) *Give an example of an* X *and a Borel* f *for which* $\sigma(Y)$ *is strictly smaller than* $\sigma(X)$*.*

Problem 4. *For an arbitrary sub-*σ*-algebra* $\mathcal{G}$*, prove the law of total expectation,* $\mathbb{E}\big[\mathbb{E}[X \mid \mathcal{G}]\big] = \mathbb{E}[X]$*.*

Problem 5. *Let* B_1, B_2, B_3 *partition* Ω *into events of positive probability, and let* $\mathcal{G} = \sigma(\{B_1, B_2, B_3\})$*. For an event* $A \in \mathcal{F}$*, compute* $\mathbb{E}[\mathbf{1}_A \mid \mathcal{G}]$*, and express your answer through the conditional probabilities* $\Pr(A \mid B_i)$*.*

Problem 6. *On the uniform probability space on* $[0, 1]$*, let* $X(u) = 3u^2$ *and*

$$Y(u) = \begin{cases} 3, & u \in [0, \frac{1}{3}), \\ u^2, & u \in [\frac{1}{3}, 1]. \end{cases}$$

Compute $\mathbb{E}[X \mid Y] := \mathbb{E}[X \mid \sigma(Y)]$*.*