Stochastic Processes 2026–27

Problem Set 2

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Instructions. You may use the axioms of a σ-algebra and of a probability measure, results proved in L01–L03, and standard rules of set theory. Recall that $X : \Omega \to \mathbb{R}$ is a random variable with respect to $\mathcal{F}$ (equivalently, $\mathcal{F}$-measurable) if $\{\omega : X(\omega) \leq b\} \in \mathcal{F}$ for every $b \in \mathbb{R}$. No other results may be assumed without proof. Venn diagrams are not proofs.

Problem 1. *Let $\mathcal{F}$ be any σ-algebra on Ω. Show that a constant function $X(\omega) = c$ is a random variable with respect to $\mathcal{F}$.*

Problem 2. *A fair coin is tossed twice, so $\Omega = \{HH, HT, TH, TT\}$.*

(a) List every σ-algebra on Ω, and state how many there are.

(b) For each, give a random variable measurable with respect to it (non-constant where possible).

(c) For each random variable in (b), determine which of the other σ-algebras it is also measurable with respect to. State the general rule that governs this.

Problem 3. *Let X, Y be random variables on a probability space $(\Omega, \mathcal{F}, \Pr)$.*

(a) Show that $-X$ is a random variable.

(b) Show that $X + Y$ is a random variable. (You may assume that $\mathbb{Q}$ is countable.)

Problem 4. *Let X be a random variable with cumulative distribution function $F_X(x) = \Pr(X \leq x)$.*

(a) Show that F_X is non-decreasing, using the axioms and results proved in class.

(b) Show that $\lim_{x\to\infty} F_X(x) = 1$. Hint: *continuity from below.*

Exploration: why we needed σ-algebras

These exercises answer why we bothered to invent σ-algebras at all. If we take the axioms of probability and drop the requirement $\Pr(\Omega) = 1$, what remains is called a *measure*: it is still non-negative and countably additive, but it may assign a set any value in $[0, \infty]$, not just values in $[0, 1]$. We write a measure as μ. Intuitively we assign each interval its length, $\mu([a, b]) = b - a$. The natural question is whether this assignment, together with the axioms of a measure, lets us measure *every* subset of $\mathbb{R}$. We show it cannot: no such measure can live on the whole power set of $\mathbb{R}$. This is exactly why we retreat to a smaller σ-algebra (the Borel sets) rather than measuring all subsets.

Throughout, for $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$ put $A + t := \{a + t : a \in A\}$, and call a measure μ *translation-invariant* if $\mu(A + t) = \mu(A)$ for every A and t.

Exercise 1. *Lebesgue measure on the Borel σ-algebra of $\mathbb{R}$ satisfies $\mu([a, b]) = b - a$ for $a \leq b$.*

(a) For an interval $[a, b]$ and $t \in \mathbb{R}$, describe $[a, b] + t$ and show $\mu([a, b] + t) = b - a$.

(b) Extend this to any finite disjoint union of intervals, and hence conclude that μ is translation-invariant. (You may assume an identity holding on all intervals extends to all Borel sets.)

Exercise 2. *Prove that there is no countably additive, translation-invariant measure μ on the power set of $\mathbb{R}$ with $\mu([a, b]) = b - a$ for all $a \leq b$. Suppose such a μ exists, and reach a contradiction through the following steps.*

(a) Show that $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$, and deduce $\mu([0, 1)) = 1$ and $\mu([-1, 2)) = 3$.

(b) On $[0, 1)$, let $x \sim y$ iff $x - y \in \mathbb{Q}$. Show $\sim$ is an equivalence relation.

(c) Let $V \subseteq [0, 1)$ contain exactly one representative of each equivalence class (Axiom of Choice), and let $q_1, q_2, \ldots$ enumerate $\mathbb{Q} \cap [-1, 1]$. Show that the sets $V + q_1, V + q_2, \ldots$ are pairwise disjoint.

(d) Show that $[0, 1) \subseteq \bigcup_n (V + q_n) \subseteq [-1, 2)$, and deduce that $1 \leq \sum_n \mu(V) \leq 3$.

(e) Show this fails whether $\mu(V) = 0$ or $\mu(V) > 0$, and conclude.