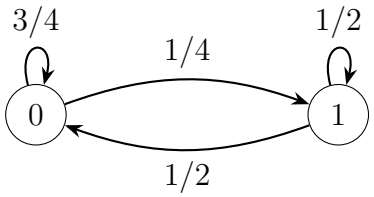
Test 2 — Stochastic Processes

Maximum Marks: 6 Duration: 30 minutes

State clearly any theorem that you use.

Notation is as in the class notes.

1. **(10 marks)** Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0,1\}$ with the following state diagram.



Define

$$Y_n = (X_n, X_{n+1}).$$

You may assume, without proof, that $(Y_n)_{n\geq 0}$ is a Markov chain.

(a) **(2 marks)** Order the states of (Y_n) as

$$00, \qquad 01, \qquad 10, \qquad 11.$$

Write down its transition probability matrix, and denote it by Q.

(b) **(4 marks)** Let P be the transition probability matrix of (X_n), and let $\pi = (\pi_0, \pi_1)$ be an invariant distribution for (X_n). Show that

$$\mu_{ij} := \pi_i P_{ij}, \qquad i,j \in \{0,1\},$$

defines an invariant distribution μ for (Y_n).

(c) **(4 marks)** Let V_N be the number of transitions from 0 to 1 among the first N transitions of (X_n). You may use the result of part (b), whether or not you have proved it. Compute

$$\lim_{N\to\infty} \frac{V_N}{N}.$$

Give a complete justification, stating clearly any theorem that you use.