Algebra of Itô Calculus

DE08: Stochastic Processes

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Quadratic covariation

For continuous processes X and Y, define

$$\boxed{[X,Y]_t := \mathbb{P}\text{-}\lim_{|\pi|\to 0} \sum_i \big(X_{t_{i+1}} - X_{t_i}\big)\big(Y_{t_{i+1}} - Y_{t_i}\big),}$$

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whenever the limit exists.
The quadratic variation previously denoted by $[X]_t$ is now written

$$\boxed{[X,X]_t.}$$

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Pass to the limits as $|\pi| \to 0$.

If $g \in C^1([0,T])$, then $[g,g]_t = 0$; hence

$$\boxed{[g,B]_t = 0.}$$

From analysis to algebra

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$[B,B]_t = t$	**Axiom** $\mathcal{I}_6$: $(dB_t)^2 = dt$

Continuous Itô processes

Let X_0 be $\mathcal{F}_0$-measurable, and let b and σ be continuous adapted processes satisfying

$$\mathbb{E}\int_0^T \left(|b_s|^2 + |\sigma_s|^2\right) ds < \infty.$$

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A process X of the form

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The processes b and σ are respectively the *drift* and *diffusion coefficient* of X.

Itô's formula

Analysis $\Longrightarrow$ **Algebra**

If $F \in C^{1,2}$ and X is a continuous Itô process, then

$$F(t, X_t) - F(0, X_0) = \int_0^t F_t(s, X_s)\, ds + \int_0^t F_x(s, X_s)\, dX_s + \frac{1}{2} \int_0^t F_{xx}(s, X_s)\, d[X, X]_s.$$

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Axiom $\mathcal{I}_7$:

$$dF(t, X_t) = F_t\, dt + F_x\, dX_t + \frac{1}{2} F_{xx} (dX_t)^2.$$

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Here

$$(dX_t)^2 = d[X, X]_t.$$

Product rule

Analysis $\Longrightarrow$ **Algebra**

Axiom $\mathcal{I}_8$:

$$\boxed{d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + dX_t\, dY_t.}$$

Theory: the Itô integral has mean zero

For an elementary adapted integrand

$$H_s = \sum_i H_i \mathbf{1}_{(t_i, t_{i+1}]}(s), \qquad H_i \in \mathcal{F}_{t_i},$$

we have

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Since $\mathbb{E}(\Delta_i B \mid \mathcal{F}_{t_i}) = 0$,

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The identity extends to square-integrable integrands by L^2 approximation.

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If H is deterministic, this reduces to

$$\boxed{\mathrm{Var}\left(\int_0^t H_s \, dB_s \right) = \int_0^t H_s^2 \, ds.}$$

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Using Axioms $\mathcal{I}_3$, $\mathcal{I}_5$, and $\mathcal{I}_6$,

$$(dY_t)^2 = (B_t\, dt + t\, dB_t)^2 = t^2\, dt.$$

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Therefore

$$\boxed{[Y, Y]_t = \int_0^t s^2\, ds = \frac{t^3}{3}.}$$

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Since Brownian motion starts at $B_0 = 0$, equivalently,

$$\boxed{B_t^2 = t + 2\int_0^t B_s\, dB_s.}$$

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Write $B_t^2 = B_t B_t$. By Axiom $\mathcal{I}_8$,

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$$\boxed{d(\log X_t) = \left(\frac{b_t}{X_t} - \frac{\sigma_t^2}{2X_t^2}\right)dt + \frac{\sigma_t}{X_t}\,dB_t.}$$

Problem 4: solve $dX_t = 5\,dt + 4\,dB_t$

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It is enough to choose

$$F_t = 5, \qquad F_x = 4, \qquad F_{xx} = 0.$$

Problem 4: construct the solution

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Since $d(e^{-5t}) = -5e^{-5t}dt$, Axioms $\mathcal{I}_3$ and $\mathcal{I}_5$ give

$$d(e^{-5t})\,dX_t = -5e^{-5t}dt\,(5X_t dt + 4dB_t) = 0.$$

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$$d(e^{-5t})\,dX_t = -5e^{-5t}dt\,(5X_t dt + 4dB_t) = 0.$$

Substituting into the product rule,

$$d(e^{-5t}X_t) = 4e^{-5t}\,dB_t.$$

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Multiplying by e^{5t},

$$\boxed{X_t = e^{5t}X_0 + 4\int_0^t e^{5(t-s)}\,dB_s.}$$

Problem 6: the Ornstein–Uhlenbeck process

For $\kappa > 0$, the OU equation is

$$\boxed{dX_t = \kappa(\theta - X_t)\,dt + \sigma\,dB_t, \qquad X_0 = x.}$$

Problem 6: the Ornstein–Uhlenbeck process

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$$\boxed{dX_t = \kappa(\theta - X_t)\,dt + \sigma\,dB_t, \qquad X_0 = x.}$$

Use the integrating factor $e^{\kappa t}$. Axiom $\mathcal{I}_8$ and the multiplication axioms give

$$d(e^{\kappa t} X_t) = \kappa\theta e^{\kappa t}\,dt + \sigma e^{\kappa t}\,dB_t.$$

Problem 6: the Ornstein–Uhlenbeck process

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Problem 7: geometric Brownian motion

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Gaussian processes

A process $(G_t)_{t\in[0,T]}$ is called a *Gaussian process* if, for every $n \geq 1$ and every $t_1, \ldots, t_n \in [0, T]$,

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The Gaussian vector is allowed to be degenerate; its covariance matrix need not be invertible.

L^2 limits of multivariate Gaussian vectors

Lemma. Let $Z_n, Z \in \mathbb{R}^d$. If every Z_n is multivariate Gaussian and

$$\mathbb{E}\|Z_n - Z\|^2 \longrightarrow 0,$$

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Thus $a^{\mathsf{T}} Z$ is an L^2 limit of scalar Gaussian variables, hence is Gaussian. Since this holds for every a, Z is multivariate Gaussian.

Deterministic Itô integrals are Gaussian processes

Claim. If $f \in L^2([0,T])$ is deterministic, then

$$\boxed{G_t := \int_0^t f(s)\, dB_s, \qquad 0 \le t \le T,}$$

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$$G_t^{(m)} = \int_0^t f_m(s)\, dB_s.$$

For fixed $t_1, \dots, t_n$, the vector $(G_{t_1}^{(m)}, \dots, G_{t_n}^{(m)})$ is multivariate Gaussian.

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By the Itô isometry,

$$\mathbb{E} \sum_{j=1}^{n} |G_{t_j}^{(m)} - G_{t_j}|^2 = \sum_{j=1}^{n} \int_0^{t_j} |f_m(s) - f(s)|^2\, ds \longrightarrow 0.$$

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The preceding lemma now shows that $(G_{t_1}, \dots, G_{t_n})$ is multivariate Gaussian. Hence G is a Gaussian process.

Example: the process in Problem 5 is Gaussian

Recall

$$X_t = e^{5t}\left(X_0 + 4\int_0^t e^{-5s}\, dB_s\right).$$

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If X_0 is deterministic, or Gaussian and independent of B, then deterministic affine transformations preserve Gaussianity.
Therefore

$$\boxed{(X_t)_{t\geq 0} \text{ is a Gaussian process.}}$$

Example: OU mean, variance, and distribution

The mean-zero property of the Itô integral gives

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The OU solution is a deterministic affine transformation of a Gaussian process. Consequently,

$$X_t \mid X_0 = x \sim \mathcal{N}\left(\theta + (x - \theta)e^{-\kappa t}, \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\right).$$

Example: OU stationarity

Assume

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which depends only on the time difference.
Since the OU process is Gaussian, its finite-dimensional distributions are time-shift invariant. Hence it is stationary.

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Although S is not Gaussian, $\log S$ is Gaussian. Using the Gaussian exponential moment,

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For $\sigma > 0$, geometric Brownian motion is not stationary:

$$\mathrm{Var}(\log S_t) = \sigma^2 t$$

grows with time.