Internal 2 — Stochastic Processes

Maximum Marks: 34 Duration: 90 minutes

Every step must be justified. State clearly any theorem that you use. Part marks are only given for steps that move in the direction of the

Notation is as in the class notes. *Advice: Dont solve all problems, you wont have time. The paper is out of 43 marks and you can drop 9 marks worth so do what you know properly.*

1. **(10 marks)** Answer each of the following in one or two precise sentences.

   (a) **(3 marks)** 2 balls are distributed between two urns A and B. At each step one of the 2 balls is selected uniformly at random and transferred to the other urn. Let X_n be the number of balls in urn A after n steps. Write the transition probability matrix of this chain.

   (b) **(3 marks)** If possible, construct a DTMC with one absorbing state and exactly one recurrent non-absorbing state. Explain your answer.

   (c) **(4 marks)** For a DTMC X and states $i, x_0, \ldots, x_n$, prove

   $$\Pr\{X_{n+2} = i \mid X_0 = x_0, \cdots, X_n = x_n\} = \Pr\{X_{n+2} = i \mid X_n = x_n\}$$

   from first principles. Use only the definition of the Markov property and basic theorems from probability theory.

2. **(6 marks)** Let a time-homogeneous DTMC have state space S and n-step transition probabilities $p_{ij}^{(n)}$.

   (a) **(1 marks)** Define what it means for two states $i, j \in S$ to communicate.

   (b) **(5 marks)** Prove that communication is an equivalence relation on S.

3. **(7 marks)** Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and let $(X_t)_{t\geq 0}$ satisfy

   $$dX_t = (2X_t + 3)\,dt + 4\,dB_t, \qquad X_0 = x.$$

   (a) **(4 marks)** Solve the SDE and express X_t in terms of x and an Itô integral.

   (b) **(3 marks)** Compute $\mathbb{E}[X_t^2 \mid X_0 = x]$.

4. **(8 marks)** Let $(S_t)_{t\geq 0}$ satisfy the geometric Brownian motion SDE

   $$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t, \qquad S_0 = s > 0,$$

   where μ and σ are constants. Assume that the solution has the form $S_t = f(t, B_t)$, where f is sufficiently smooth.

(a) **(4 marks)** Use Itô's formula and compare the dt and dB_t terms to show that f satisfies

$$f_t + \frac{1}{2} f_{xx} = \mu f, \qquad f_x = \sigma f, \qquad f(0,0) = s.$$

(b) **(4 marks)** Solve the second equation first and substitute the result into the first equation. Hence find $f(t, B_t)$ explicitly.

5. **(6 marks)** Let $(B_t)_{t \geq 0}$ be a standard Brownian motion. Prove that, for $T \geq 0$,

$$\int_0^T (B_t^3 - 3tB_t)\, dB_t = \frac{1}{4} B_T^4 - \frac{3}{2} T B_T^2 + \frac{3}{4} T^2.$$

You may use Itô's formula.

6. **(6 marks)** A bond is rated good (G), risky (R), or in default (D); default is absorbing. Each year the rating changes according to

$$P = \begin{pmatrix} 0.88 & 0.10 & 0.02 \\ 0.08 & 0.82 & 0.10 \\ 0 & 0 & 1 \end{pmatrix},$$

in the order (G, R, D).

(a) **(2 marks)** Identify the transient and absorbing states. Starting from G and from R, find the probability of default within two years.

(b) **(2 marks)** Find the expected time to default, starting from G.

(c) **(2 marks)** A two-year zero-coupon bond pays Rs. 100 at the end of year 2 if it has not defaulted by then, and Rs. 0 otherwise. With an annual risk-free rate of 5%, compute its present value if it is currently rated G.