Internal 1 — Stochastic Processes

Maximum Marks: 34

Every step must be justified from the axioms of a σ-algebra, the axioms of probability, the defining properties of conditional expectation, a theorem proved in class, or an explicit logical argument. Steps carry marks, not the final answer.

1. **(12 marks)** *(First Borel–Cantelli Lemma.)* Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space and let $(A_n)_{n\geq 1}$ be a sequence of events with $\sum_{n=1}^{\infty} \Pr(A_n) < \infty$. Prove that

$$\Pr\left(\limsup_{n\to\infty} A_n\right) = 0.$$

   You may use, stating where you invoke it, that $\limsup_n A_n = \bigcap_{k\geq 1} \bigcup_{l\geq k} A_l$ and that this set lies in $\mathcal{F}$; every other step must be justified from the axioms of probability.

2. **(10 marks)**

   (a) **(2 marks)** State the definition of the conditional expectation $\mathbb{E}[X \mid \mathcal{G}]$ of an integrable random variable X given a sub-σ-algebra $\mathcal{G}$ (the two defining axioms).

   (b) **(2 marks)** When $\mathcal{G}$ is generated by the atoms $C_1, \ldots, C_m$ (a partition of Ω with $\Pr(C_j) > 0$), write down the formula for $\mathbb{E}[X \mid \mathcal{G}]$, explaining every symbol in it.

   (c) **(6 marks)** On the uniform probability space $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, $\Pr([a,b]) = b-a$, let $X(u) = 4u^3$ and

$$Y(u) = \begin{cases} 0, & u \in [0, \frac{1}{3}], \\ 1, & u \in (\frac{1}{3}, \frac{2}{3}], \\ 2, & u \in (\frac{2}{3}, 1]. \end{cases}$$

   Compute $\mathbb{E}[X \mid Y] := \mathbb{E}[X \mid \sigma(Y)]$.

3. **(8 marks)** A pensioner receives \$2 (thousand) at the beginning of each month. The amount he spends during a month is independent of the amount he holds and equals i with probability $P_i = \frac{1}{4}$, for $i = 1, 2, 3, 4$. If at the end of a month he holds more than \$3, he gives the excess over \$3 to his son.

   (a) **(4 marks)** Let X_n be the amount of capital the pensioner holds at the *end* of month n (after spending, and after giving any excess over \$3 to his son). This is a Markov chain; identify its states and write down the transition probability matrix.

   (b) **(4 marks)** Suppose $X_0 = 3$ (just after his payment at the start of month 1 he holds \$5). Find the expected capital at the end of month 3 given the entire history of his capital up to the end of month 2, i.e. $\mathbb{E}[X_3 \mid X_0, X_1, X_2]$.

4. **(8 marks)**

(a) **(4 marks)** Let $\{M_n\}$ be a martingale adapted to the filtration $\{\mathcal{F}_n\}$. Prove that $\mathbb{E}[M_{n+2} \mid \mathcal{F}_n] = M_n$.

(b) **(4 marks)** Define a harmonic function on a Markov chain.