Stochastic Process Week 7 HW

Srikanth Pai, Asst. Prof, MSE, Chennai

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1. **A two-period consumption–investment decision.**

   A farmer begins with one unit of grain. In the first period he faces a choice: either consume the grain immediately, yielding utility 1, or plant it. If planted, the grain produces 2 units in the second period, which must then be consumed, yielding utility 2. There is no discounting.

   (a) Define the state space, the set of available actions, the transition rule, and the associated reward function.

   (b) Write the Bellman equations for periods 2 and 1.

   (c) Solve the Bellman equations by backward induction and compute the value of each available action in period 1.

   (d) Identify the optimal policy and the associated value function.

2. **An inventory control problem with stochastic demand.**

   A store manages a small inventory that can take only three levels: zero, one, or two units. At the beginning of each period, a random demand for at most one unit arrives. With probability p the demand is one unit, and with probability $1-p$ the demand is zero. If a unit is demanded and inventory is available, the store makes a sale and receives revenue equal to one. If there is no demand, or if demand occurs but no inventory is available, no sales are made. After demand has been observed, the store may replenish by ordering at most one unit at a cost $c \in (0,1)$. Inventory can never exceed two units. Future payoffs are discounted by the factor $\beta \in (0,1)$.

   Formally, let $s \in \{0,1,2\}$ denote the inventory level at the start of a period, and let $d \in \{0,1\}$ denote demand, with $\mathbb{P}(d=1)=p$. Sales revenue is $\min\{s,d\}$. After demand is realized, the decision variable $a \in \{0,1\}$ represents the order placed, subject to the inventory cap $\min\{s-d+a,2\}$. The immediate payoff in a period is $\min\{s,d\}-ca$, and the continuation value is given by the discounted value function $V(\cdot)$. Denote $V_s = V(s)$.

   The objective is to characterize the threshold ordering cost $c^*(p,\beta)$ such that it is optimal to place an order when $s=1$ after a no-sale if and only if $c \leq c^*(p,\beta)$.

   (a) Write the Bellman equations for V_0, V_1, V_2.

(b) Show that $V_2 \geq V_1 \geq V_0$, and deduce that ordering when $s = 2$ is never optimal.

(c) Reduce the Bellman equations accordingly and consider two candidate policies at $s = 1$: always order, or never order.

(d) Solve the Bellman system under each policy to obtain closed-form expressions for V_0, V_1, V_2.

(e) Derive the optimality condition for ordering at $s = 1$ after a no-sale:

$$-c + \beta V_2 \;\geq\; \beta V_1.$$

(f) By substituting the values under the "do not order" policy and solving the equality case, obtain the explicit rational function

$$c^*(p, \beta) \;=\; \frac{p^2\beta^2}{1 - \beta + 2p\beta - 2p\beta^2 + p^2\beta^2}.$$

(g) State the optimal policy rule in terms of c and $c^*(p, \beta)$.

3. **A job search problem.**

Each period a worker receives a wage offer $W_t \sim \text{Uniform}(0, 1)$, independently across t. If she accepts an offer w, she works forever at that wage and obtains discounted utility $w/(1 - \beta)$. If she rejects, she receives 0 for that period and moves on to the next offer.

(a) Formulate the problem as a Markov decision process, specifying states, actions, and payoffs.

(b) Write the Bellman equation for the value function V.

(c) Show that the optimal policy is characterized by a reservation wage w^*: the worker accepts if and only if $W_t \geq w^*$.