HW 5: Stoc Proc 25-26

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1 Definitions

Definition 1 (Open and Closed Communicating Classes)**.** *Let $\{X_n\}$ be a discrete-time Markov chain with state space S and transition probability matrix P.*

- *A communicating class $C \subseteq S$ is* ***closed*** *if for all $i \in C$ and $j \notin C$, we have $p_{ij} = 0$. That is, once the chain enters a closed class, it cannot leave.*
- *A communicating class $C \subseteq S$ is* ***open*** *if it is not closed. That is, there exist $i \in C$ and $j \notin C$ such that $p_{ij} > 0$.*

Definition 2 (Irreducible)**.** *A discrete-time Markov chain is* ***irreducible*** *if its state space forms a single communicating class. Equivalently, for any two states $i, j \in S$, there exist positive integers m, n such that $p_{ij}^{(m)} > 0$ and $p_{ji}^{(n)} > 0$.*

2 Classification Problem

Problem 1. *Consider a Markov chain with 6 states representing different phases of a system: $\{1, 2, 3, 4, 5, 6\}$. The transition probability matrix is:*

$$P = \begin{pmatrix} 0.5 & 0.3 & 0.2 & 0 & 0 & 0 \\ 0.4 & 0.4 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0.3 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0.9 \end{pmatrix}$$

Identify all communicating classes and determine whether each class is open or closed. Provide rigorous proof for your classification by:

1. *Determining which states communicate with each other*
2. *Identifying the communicating classes*
3. *For each class, checking if there are any transitions out of the class*
4. *Classifying each class as open or closed based on the definition*

3 Key Theorems

Theorem 1 (Class Property of Transience/Recurrence)**.** *Transience and recurrence are class properties. That is, if states i and j communicate, then either both are transient or both are recurrent.*

Theorem 2. *If $f_{ij} = 1$ and state i is recurrent, then state j is also recurrent.*

Theorem 3. *Any finite closed communicating class is positive recurrent.*

Theorem 4. *Any open communicating class is transient.*

4 Application Problems

4.1 Instructions for problems below

For each application problem:

- Clearly define the state space
- Draw the state transition diagram
- Identify and classify all communicating classes
- Apply the theorems to determine transience/recurrence
- Perform the requested calculations for expected visits or return times

Show all work and **justify your reasoning using the definitions and theorems provided in section 3**.

4.2 Problem statements

Problem 2. *A cereal company includes one of 3 different collectible cards (A, B, C) in each box with equal probability 1/3. A collector has a superstition: they want to end their collection by drawing card A as their final card.*
Their strategy is:

Keep collecting until they have all 3 cards AND their most recent draw was card A.

If they complete the set A,B,C but their final card was B or C, they continue collecting (hoping to draw A again to "properly" end) They only stop when they draw card A while already having cards B and C

Model this as a Markov chain where the state represents the set of cards collected. Also create an end state. Identify all transient and recurrent states. For the transient states, compute the expected number of times each state is visited starting from the empty set. For recurrent states, compute the expected return time.

Problem 3. *A small store manages inventory for a product with the following policy:*

- *If inventory level is 0 at the end of a day, order 3 units (available next day)*
- *If inventory level is 1 at the end of a day, order 2 units (available next day)*
- *If inventory level is 2 at the end of a day, order 0 units*
- *If inventory level is 3 at the end of a day, order 0 units*

Daily demand follows the pattern:

- *Demand 0: probability 0.1*
- *Demand 1: probability 0.4*

- *Demand 2: probability 0.3*
- *Demand 3: probability 0.2*

Unsatisfied demand is lost. Model this as a Markov chain with states representing end-of-day inventory levels $\{0, 1, 2, 3\}$*. Determine the transition probability matrix and classify all states as transient or recurrent.*

Problem 4 (Modified Gambler's Ruin)**.** *A gambler starts with \$2 and plays a game where the bet size depends on their current wealth:*

- *If wealth* $\leq$ *\$2: bet \$1 (win/lose \$1 with probability 0.4/0.6)*
- *If wealth* $=$ *\$3: bet \$2 (win/lose \$2 with probability 0.4/0.6)*
- *If wealth* $\geq$ *\$4: bet \$3 (win/lose \$3 with probability 0.4/0.6)*

The game continues until the gambler reaches \$0 (bankruptcy) or more than or equal to \$7 (target wealth).

This creates the following transitions:

- *From \$1: go to \$0 (prob 0.6) or \$2 (prob 0.4)*
- *From \$2: go to \$1 (prob 0.6) or \$3 (prob 0.4)*
- *From \$3: go to \$1 (prob 0.6) or \$5 (prob 0.4)*
- *From \$4: go to \$1 (prob 0.6) or \$7 (prob 0.4)*
- *From \$5: go to \$2 (prob 0.6) or \$8 (prob 0.4)*
- *From \$6: go to \$3 (prob 0.6) or \$9 (prob 0.4)*

Model this as a Markov chain with states $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *representing the gambler's current wealth. Classify all states. For transient states, find the expected number of visits starting from state 2.*