Week 3 HW, Stoc. Proc 25-26

Srikanth Pai, MSE

August 3, 2025

1. Consider a DTMC with state space $\{1, 2, 3\}$ and transition matrix:

$$P = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}$$

   (a) If $P_0 = (0.4, 0.4, 0.2)$ is the initial distribution on the states $(1, 2, 3)$, compute $P(X_0 = 1, X_1 = 2, X_2 = 3)$.

   (b) Find the two-step transition matrix P^2.

   (c) Compute $\mathbb{E}\left(X_3 | X_0\right)$. Recall that this conditional expectation is a function of X_0.

2. For the chain in problem 1, determine which of the following are stopping times with proof:

   (a) $T_1 = \min\{n \geq 0 : X_n = 3\}$.

   (b) $T_2 = \min\{n \geq 0 : X_{n+1} = 1\}$.

   (c) $T_3 = \min\{n \geq 0 : X_n = 1 \text{ and } X_{n-1} = 3\}$ (with $T_3 = \infty$ if $n = 0$).

3. For the random time $S = \max\{n \leq 2 : X_n = 2\}$:

   (a) PROVE S is not a stopping time.

   (b) Starting from $X_0 = 2$, calculate:

      - $P(X_{S+1} = 3 | X_S = 2, S = 0)$
      - $P(X_{S+1} = 3 | X_S = 2, S = 2)$
      - $P(X_{S+1} = 3 | X_S = 2)$

      Show these are different.

4. Consider the chain with transition matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0 & 0.2 & 0.5 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   (a) Which states are transient/recurrent?

   (b) Let $T_{24} = \min\{n \geq 1 : X_n = 4 | X_0 = 2\}$. Calculate $f_{24}^{(1)}$, $f_{24}^{(2)}$, $f_{24}^{(3)}$ and f_{24}.

   (c) Find the mean return time $\nu_{22} = E[T_{22}]$.

5. Show that if S,T are stopping times, then show that minS,T, maxS,T, S+T are all stopping times. Show an example where S-T is not a stopping time.

6. Let $\{X_n : n \geq 1\}$ is an independent and identically distributed (i.i.d.) sequence $\mathbb{E}(|X|) < \infty$. We are interested in the sum of the r.v.s. up to time n,

$$S_n = X_1 + \cdots + X_n.$$

If T is a stopping time with $\mathbb{E}(T) < \infty$, **prove** that

$$\mathbb{E}(S_T) = \mathbb{E}(X)\mathbb{E}(T).$$

Let $\{S_n | n \geq 0\}$ be a symmetric random walk starting at 0. Let $N = \inf\{n \in \mathbb{N} : S_n \in (-a, b)\}$ where $a, b > 0$ are fixed integers. Compute $\Pr\{S_N = b\}$. [Hint: Show $\mathbb{E}(S_N) = 0$ and then compute the expectation using definition.]