25-26, Srikanth Pai, MSE

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1. Prove that a sequence of discrete iid random variables is a discrete time markov chain.
2. (IMP) Prove the two facts stated in class about DTMC.
3. (IMP) If $A_1, A_2, A_3, ..$ is a sequence of given events and B is the event that infinitely many of the given events occur,
    (a) Show that $B = \bigcap_{i=1}^{\infty} \bigcup_{n=i}^{\infty} A_n$. We will usually denote B as "A_n infinitely often".
    (b) If $\sum_{i=1}^{\infty} P(A_n) < \infty$, then show that $P(A_n \text{ infinitely often }) = 0$. [Hint: Use union bound and Kolmogorov equivalent axiom 3 proved in last homework.]
4. Two urns, A and B, hold a total of 10 balls each. The first one has only red balls and the second one has only white balls. At each step, pick a ball at random from each box and move it to the other urn. You repeat this step forever. Describe a DTMC for the state of the system. Write the probability transition matrix. Write a *python program* to compute P^{1000}, P^{1001}. Interpret the results.
5. A gambler starts with $10k$ rupees and bets $1k$ rupees per game, with probability p of winning each game. He has a goal of reaching $15k$ without going broke. Model the state of the gambler as a DTMC. Write the tpm. What is the probability of reaching 13k rupees in exactly five steps?
6. Let (X_n) be a sequence of coin tosses of a biased coin with probability of heads as p. Let R_n denote the current run of heads up to time n. For example, for the coin toss sequence 1100010011100, the run sequence is $R = [1, 2, 0, 0, 0, 1, 0, 0, 1, 2, 3, 0, 0]$. PROVE that (R_n) is a DTMC.
7. A store maintains inventory with a fixed base-stock level $S \in \mathbb{N}$. At each day $t = 0, 1, 2, \ldots$, the distribution of demand $D_t \sim \text{Uniform}(S + 1)$, i.i.d. across time. The inventory is restocked every day to maintain base stock level. PROVE that the inventory level is a DTMC. Write the tpm For $S = 3$.
8. A borrower is making monthly payments on a loan. Let the borrower's status at time step t be the number of months they are behind on their loan payments. This number can range from zero (completely up to date) up to a fixed maximum B, at which point the borrower is declared to be in default. Once in default, they remain there permanently. At each time step:
    (a) With a fixed probability p, the borrower makes a successful payment. If they are already behind on payments, this reduces their delay by one month. If they are fully caught up (zero months behind), nothing changes.
    (b) With the remaining probability 1 - p, the borrower fails to make the payment. This increases their delay by one month, unless they are already at the default limit B, in which case their delay remains unchanged.

    Construct a DTMC for this problem.