Week 15

Srikanth Pai, Asst. Prof, MSE

1. Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate $\lambda = 0.5$.
   (a) Find the probability of no arrivals in $(3, 5]$.
   (b) Find the probability that there is exactly one arrival in each of the following intervals: $(0, 1]$, $(1, 2]$, $(2, 3]$, and $(3, 4]$.

2. Compute $\mathbb{E}\left(N(t_1) \mid N(t_2)\right)$.

3. Let $\boldsymbol{T}^{(n)} = (T_1, T_2, \cdots, T_n)$ the vector of n-arrival times. Let $\mathrm{f}_{\boldsymbol{T}^{(n)}}\left(\boldsymbol{t}^{(n)}\right)$ be the joint density of $\boldsymbol{T}^{(n)}$. Prove that this density is, over the region $0 < t_1 < \cdots < t_n$, has the value

$$\mathrm{f}_{\boldsymbol{T}^{(n)}}\left(\boldsymbol{t}^{(n)}\right) = \lambda^n e^{-\lambda t_n}$$

   [Hint: We proved this for n=2 in class.]

4. Let $\boldsymbol{T}^{(n)} = (T_1, T_2, \cdots, T_n)$ the vector of n-arrival times. Let $\mathrm{f}_{\boldsymbol{T}^{(n)}|N(t)}\left(\boldsymbol{t}^{(n)} \mid n\right)$ be the joint density of $\boldsymbol{T}^{(n)}$ conditional on $N(t) = n$. Prove that this density is constant over the region $0 < t_1 < \cdots < t_n < t$ and has the value

$$\mathrm{f}_{\boldsymbol{T}^{(n)}|N(t)}\left(\boldsymbol{t}^{(n)} \mid n\right) = \frac{n!}{t^n}$$

   In summary, this means that conditioning on $N(t) = n$, the n arrival epochs are the order statistics of n i.i.d. $U(0, t)$. [Hint: Use previous problem along with Bayes theorem.]

5. Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate λ, and T_i be the ith arrival time for some natural number i. For a natural number n, compute

$$P\left(T_i \leq x \mid N(t) = n\right)$$

   [Hint: You can use previous problem or do it from first principles. Both methods are worth learning.]

6. Let $\boldsymbol{T}^{(n)} = (T_1, T_2, \cdots, T_n)$ the vector of n-arrival times.
   (a) Compute $Cov(T_1, T_2)$.
   (b) Compute $Cov\left(T_1, T_2 \mid N(t) = 2\right)$.

7. Recall that the moment generating function of a normal distribution with mean μ and variance σ^2 is given by

$$M(u) = e^{u\mu + \sigma^2 u^2/2}.$$

   Compute the second, third and fourth moments of a normal random variable.

8. Let $W(t)$ be a Wiener process,

   (a) Prove $W(t)$ is a martingale.

   (b) Prove $W(t)^2 - t$ is a martingale.

   (c) Prove $W(t)^3 - 3tW(t)$ is a martingale.

   (d) Find coefficients $a(t), b(t)$ so that $W(t)^4 + a(t)W(t)^2 + b(t)$ is a martingale.

9. Show that if $p(t,x)$ satisfies the heat equation, then so does

$$q(t,x) = \frac{1}{\sqrt{\lambda}}\, p\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), \qquad \lambda > 0.$$

   Interpret this result in probabilistic terms as the Brownian scaling property $W_t \stackrel{d}{=} \lambda W_{t/\lambda^2}$. [Hint: Sketched in class.]

10. Prove or disprove: If $W_1(t), W_2(t)$ are independent Wiener processes, then $\sqrt{k}W_1(t) + \sqrt{1-k}W_2(t)$ is also a Wiener process.

11. (Hitting Times for Brownian Motion) Let $W(t)$ be a standard Brownian motion. Let $a > 0$. Define T_a as the first time that $W(t) = a$ for some real number a. That is

$$T_a = \min\{t : W(t) = a\}$$

   (a) Show that for any $t \geq 0$, we have

$$P(W(t) \geq a) = P(W(t) \geq a \mid T_a \leq t)\, P(T_a \leq t)\,,$$

   where $T_a = \inf\{s \geq 0 : W(s) = a\}$.

   (b) Using Part (a), show that

$$P(T_a \leq t) = 2\left[1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right].$$

   (c) Using Part (b), show that the PDF of T_a is given by

$$f_{T_a}(t) = \frac{a}{t\sqrt{2\pi t}} \exp\left(-\frac{a^2}{2t}\right), \qquad t > 0.$$