Week 10 HW - Martingales and Conditional expectations

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1. Let $(\xi_n)_{n\geq 1}$ be i.i.d. random variables with $\mathbb{P}(\xi_n = 1) = \mathbb{P}(\xi_n = -1) = 1/2$. Define the simple symmetric random walk

$$S_n = \xi_1 + \cdots + \xi_n, \quad n \geq 1, \qquad S_0 = 0.$$

   (a) Show that $(S_n, \mathcal{F}_n)$ with $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$ is a martingale.
   (b) Show that $(S_n^2 - n)$ is also a martingale.
   (c) Use the tower property of conditional expectation to compute $\mathbb{E}[S_{n+m} \mid \mathcal{F}_n]$.

2. With (S_n) as above, fix $\lambda > 0$ and consider the process

$$M_n = \exp\{\lambda S_n - n\psi(\lambda)\}, \quad \psi(\lambda) = \log \mathbb{E}[e^{\lambda \xi_1}].$$

   (a) Prove that (M_n) is a martingale.
   (b) Conclude that $(e^{\lambda S_n})$ is a submartingale for any $\lambda > 0$.

3. Let X be uniformly distributed on $\{1, 2, 3, 4\}$, and let $Y = \mathbf{1}_{\{X \text{ is even}\}}$.

   (a) Find $\sigma(Y)$ and describe its atoms.
   (b) Compute $\mathbb{E}[X \mid \sigma(Y)]$ explicitly.

4. Let X, Y be independent fair coin tosses taking values in $\{0, 1\}$.

   (a) Describe explicitly the atoms of $\sigma(X, Y)$.
   (b) Compute $\mathbb{E}[X + Y \mid \sigma(X)]$ and $\mathbb{E}[X + Y \mid \sigma(Y)]$.
   (c) Compute $\mathbb{E}[X + Y \mid \sigma(X + Y)]$ and explain why this is the conditional expectation onto the partition $\{X + Y = 0, 1, 2\}$.

5. Let (S_n) be a biased random walk with $\mathbb{P}(\xi_i = 1) = p \neq 1/2$, $\mathbb{P}(\xi_i = -1) = 1 - p$, and $S_n = \xi_1 + \cdots + \xi_n$.

   (a) Show that (S_n) is a submartingale if $p > 1/2$, a supermartingale if $p < 1/2$.
   (b) Let (H_n) be a bounded predictable sequence of bets. Define $M_n = \sum_{k=1}^{n} H_k(\xi_k)$. Show that (M_n) is again a submartingale (if $p > 1/2$) or supermartingale (if $p < 1/2$).

6. Let $\Omega = \{1, 2, 3, 4\}$ with probabilities

$$\mathbb{P}(\{1\}) = \tfrac{1}{2}, \quad \mathbb{P}(\{2\}) = \tfrac{1}{4}, \quad \mathbb{P}(\{3\}) = \mathbb{P}(\{4\}) = \tfrac{1}{8}.$$

   Define

$$X = \mathbf{1}_{\{1\}}, \qquad \sigma(Y) = \{\{1,2\}, \{3,4\}\}, \qquad \sigma(Z) = \{\{1,3\}, \{2,4\}\}.$$

   (a) Compute $\mathbb{E}[X \mid \sigma(Y)]$ and then $\mathbb{E}[\mathbb{E}[X \mid \sigma(Y)] \mid \sigma(Z)]$.
   (b) Compute $\mathbb{E}[X \mid \sigma(Z)]$ and then $\mathbb{E}[\mathbb{E}[X \mid \sigma(Z)] \mid \sigma(Y)]$.
   (c) Show explicitly that the two results differ (e.g. at $\omega = 4$), hence

$$\mathbb{E}[\mathbb{E}[X \mid \sigma(Y)] \mid \sigma(Z)] \neq \mathbb{E}[\mathbb{E}[X \mid \sigma(Z)] \mid \sigma(Y)].$$

   (d) Conclude: the tower property requires one sigma-algebra to be contained in the other; without nesting, iterated conditionals are not symmetric.

7. Suppose (Z_n) are i.i.d. with mean zero and variance one. Define

$$X_n = \sigma_n Z_n, \qquad \sigma_n^2 = \alpha_0 + \alpha_1 X_{n-1}^2 + \beta_1 \sigma_{n-1}^2, \quad n \geq 1,$$

   with positive constants $\alpha_0, \alpha_1, \beta_1$ and starting $\sigma_0^2 > 0$.

   (a) Show that $\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] = 0$.
   (b) Show that $\mathbb{E}[X_n^2 \mid \mathcal{F}_{n-1}] = \sigma_n^2$.

   Remark. The recursion

$$\sigma_n^2 = \alpha_0 + \alpha_1 X_{n-1}^2 + \beta_1 \sigma_{n-1}^2$$

   is the standard *GARCH(1,1)* model (ARCH(1) if $\beta_1 = 0$). The condition $\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] = 0$ makes (X_n) a martingale difference sequence: given past information, the expected next return is zero, so its sign cannot be predicted. At the same time σ_n^2 is $\mathcal{F}_{n-1}$–measurable, so the size of the next move (its variance) can be forecast from past data. This is the phenomenon of *volatility clustering*: large shocks are likely to be followed by large shocks, small shocks by small shocks, even though the direction of the next shock remains unpredictable. This captures the intuition of the Efficient Market Hypothesis: no predictable profits from returns, but predictable risk through time-varying volatility.

8. The *Permanent Income Hypothesis* (Friedman, 1957) states that households smooth consumption over time, reacting mainly to changes in permanent rather than transitory income. Hall (1978) showed that under rational expectations and quadratic utility, this implies the *random walk hypothesis of consumption*: expected changes in consumption are zero given current information. Quadratic utility is a convenient assumption because it yields linear Euler equations and ensures that marginal utility is proportional to consumption.

   Suppose a representative agent has quadratic utility and faces the intertemporal budget constraint

$$c_t + a_{t+1} = (1 + r)a_t + y_t,$$

   where c_t is consumption, a_t assets, r the constant interest rate, and y_t income. Assume rational expectations and that the interest rate equals the subjective discount rate.

(a) Show that the Euler equation implies

$$\mathbb{E}[c_{t+1} \mid \mathcal{F}_t] = c_t.$$

(b) Conclude that $(c_t, \mathcal{F}_t)$ is a martingale.

(c) Interpret: why does this mean that, under the permanent income hypothesis, changes in consumption are unpredictable?

(d) What does this imply for forecasting: should $c_{t+1} - c_t$ be correlated with information in $\mathcal{F}_t$?