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1. Define a sigma-algebra. Count the number of distinct sigma-algebras on the set $\Omega = \{1,2,5,9\}$.
2. Let $\mathcal{F}$ denote the Borel sigma algebra on the real line. Show that $\{0\}$ is an element of the sigma algebra.
3. State Kolmogorov's axioms for a probability measure on a sigma algebra.
4. (Very Important Fact) Prove the following property for a probability measure:
   If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is a sequence of events, show that
   $$\lim_{n\to\infty} \Pr(A_n) = \Pr\left(\bigcup_{i=1}^{\infty} A_i\right).$$
   In fact, this property is equivalent to Kolmogorov's third axiom given the first two. So this axiom can replace axiom 3.
5. Define a cumulative distribution function. Suppose the sample space of an experiment is the set of all real numbers. Given a probability measure P on the sigma algebra generated by the intervals of type $(-\infty, b)$, define a function
   $$F(b) = P[(-\infty, b)]$$
   Prove that F is a cdf.
6. Prove or disprove whether the following functions are cdfs
   (a) $F(x) = 1 - e^{-x}$
   (b) $F(x) = \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1} x\right)$
   (c) $F(t) = \frac{1}{2} + \frac{1}{\pi}\left[\frac{\left(\frac{t}{\sqrt{3}}\right)}{\left(1+\frac{t^2}{3}\right)} + \arctan\left(\frac{t}{\sqrt{3}}\right)\right]$
7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for all x in domain,
   $$f(x) \geq 0$$
   Prove that
   $$F(b) := \int_{-\infty}^{b} f(x)dx$$
   is a differentiable cdf. [Hint: Make Poorna ma'am proud!]