Internals-I
Stochastic Process 25-26

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Maximum Marks: 40 **Time:** 90 minutes

1 Problems

1. (2 marks each) Answer the following questions (no proofs required for any of them):
   (a) A transition probability matrix has all entries equal to 0.5. How many states does this Markov chain have?
   (b) Construct a Markov chain with two states where all states are absorbing.
   (c) Let $\{X_n : n \geq 0\}$ be a stochastic process on natural numbers. Let T be the first time when the process hits an even number. For the sample path $(X_0, X_1, X_2, \ldots) = (1, 5, 7, 8, 9, 3, 4, 5, 6, \ldots)$, what is the value of T?
   (d) A Markov chain $\{X_n : n \geq 0\}$ on states $\{0, 1\}$ has the following transition probabilities $p_{10} = p_{01} = 1$. What is the value of $\Pr\{X_{2024} = 1 \mid X_0 = 0\}$?
   (e) Define stopping time for a stochastic process.
   (f) **List** all the distinct sigma-algebras on the set $\Omega = \{1, 2, 3\}$.
2. (3 marks) Given an example of a discrete time stochastic process on a discrete set that is not a Markov chain. Show that Markov property fails for your process.
3. (5 marks) Prove any ONE of the following theorems:
   (a) **Theorem** If states i and j communicate in a Markov chain, and state i is transient, then state j is also transient.
   You may assume the following FACT:
   A state i is transient if and only if
   $$\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty,$$
   where $p_{ii}^{(n)}$ is the n-step transition probability from state i to itself.
   (b) State and prove the Chapman-Kolmogorov theorem for a time-homogenous discrete time markov chain.
4. An economy alternates between two regimes:
   - State 1: Expansion,
   - State 2: Recession.

   The regime follows a two-state DTMC with transition matrix
   $$P = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}, \qquad 0 < p, q < 1.$$

(a) (1 marks) For what values of p, q is the tpm doubly stochastic?

(b) (2 marks) Show that the chain is irreducible and argue that both states are positive recurrent using a theorem.

(c) (3 marks) Starting in recession ($X_0 = 2$), compute the expected length of the current recession spell (the number of consecutive periods in state 2, including the current one).

(d) (4 marks) Starting from recession, suppose $R(n)$ is the number of times the economy is in recession in the discrete time interval $[0, n)$. Note that $R(n)$ is a random variable. Compute the average long run fraction of the time the economy is in recession, i.e. compute

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{R(n)}{n}\middle| X_0 = 2\right).$$

Justify your steps.

5. Manish and Medha bet 1 hundred rupees in each round. In other words, Medha wins 1 hundred rupees each round with probability $p = 0.5$ and loses 1 hundred rupees with probability $(1 - p) = 0.5$. Assume that Manish starts the game with 1 hundred rupees and Medha starts with 2 hundred rupees. The game ends when one of them goes bankrupt. Let M_n denote Medha's fortune (in units of hundred rupees) at the end of round n and let M_0 denote Medha's initial fortune.

(a) (3 marks) Assume that M_n is a DTMC. Draw the state diagram of the Markov chain.

(b) (3 marks) Compute f_{23}, f_{32} using recursive methods.

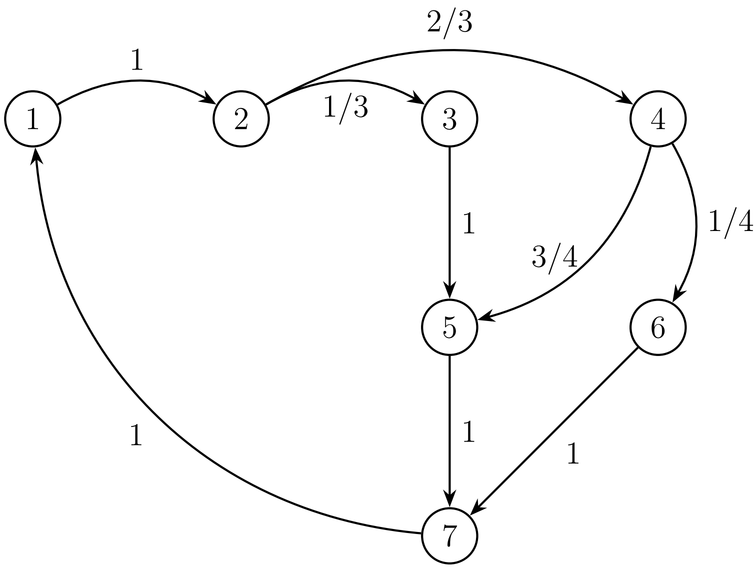
(c) (4 marks) Let T be the number of rounds played. Let $u_1 = E(T \mid M_0 = 1)$ and $u_2 = E(T \mid M_0 = 2)$. Using the law of total expectation, show that

$$u_1 = pu_2 + 1 \qquad (1)$$
$$u_2 = (1 - p)u_1 + 1 \qquad (2)$$

Carefully write the steps with justification. Now compute $E(T)$ using u_1 and u_2.

2 Bonus question

Let $\{X_t\}$ be a Markov chain with state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ with the following directed graph:



Answer the following questions:

1. (2 mark) Does the Markov chain have an invariant vector π with positive entries? If yes, compute it.

2. (2 mark) Is the state 1 periodic? Justify.

3. (2 mark) Does the $\lim_{n\to\infty} P^n_{1y}$, for each $y \in S$, exist? If it does, what is the limit?

4. (2 mark) Does the

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P^n(1,y)}{n}$$

, for each $y \in S$, exist? If it does, what is the limit?