MADRAS SCHOOL OF ECONOMICS

UNDERGRADUATE PROGRAMME IN ECONOMICS (HONOURS) [2023-26]

SEMESTER 5 [JULY – NOVEMBER, 2025]

REGULAR EXAMINATION, NOVEMBER 2025

Course Name: Stochastic Process **Course Code:** DE13

Duration: 2 Hours **Maximum Marks: 60**

Instructions: *For part A write short answers (most preferably a single word,/single number/single sentence). For part B, writing relevant formulae and quoting correct definitions helps me give you part marks. For questions with qualitative answers, creative writing is prohibited.*

Part A: Answer all questions (1 mark x 10 questions = 10 marks)

1. If $X = 4$ almost surely, what is the expectation of X^2?
2. Prove that the stochastic process $X_n = 1$ for all natural n, is a martingale.
3. Define a stationary stochastic process.
4. State the theorem that guarantees that a gambler cannot beat a fair game by sizing or timing their bets.
5. Is it possible that the mean number of visits to a given state in an irreducible DTMC is finite? Explain
6. Let $N(t)$ be a Poisson process, then what is the value of $\mathbb{E}(N(2t)^2)$?
7. If $P(A) = 0.4$, $\mathbb{E}[X1_A] = 3.2$, find $\mathbb{E}[X|A]$.
8. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and let
$$\mathcal{F} = \sigma(\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}).$$
Define $X(\omega_1) = X(\omega_2) = 1$ and $X(\omega_3) = X(\omega_4) = 2$. Show that X is measurable with respect to $\mathcal{F}$.
9. Let $T = \inf\{t \geq 0 : W(t) = 2 - 4t\}$, where $\{W(t)\}$ is Wiener process. Assume that the Optional Stopping Theorem applies for T and $W(t)$. Find $\mathbb{E}[T]$.
10. Write down the Bellman equation for a MDP. Make sure to define all your variables.

Part B: Answer any five. (5 mark x 10 questions = 50 marks)

11. Answer each subpart in not more than 100 words (be succinct and highlight stochastic concepts):

    (a) [5 marks] Explain the use of markov chains in Monte Carlo simulation with relevant equations and stating the relevant theorems done in class to justify your arguments.

    (b) [5 marks] Briefly explain the idea of game-theoretic probability proposed by Shafer and Vovk. How does the concept of forcing replace measure-based probability, and what does it mean to "force" the law of large numbers?

12. Answer the following questions:

    (a) A stock currently costs \$100. Its one-period return depends on the previous market move, modeled as a two-state Markov chain with states U (up) and D (down). The risk-free rate is $r = 2\%$ per period. If the current state is $s \in \{U, D\}$, then in the next period the stock either goes up by a factor u_s or down by a factor d_s, according to the following table:

State s	u_s	d_s
U	1.20	0.95
D	1.05	0.85

    Let q_s denote the risk-neutral probability of an up move given the current state s. The process of market states $\{M_t\}$ thus forms a Markov chain with transition probabilities $\Pr(M_{t+1} = U \mid M_t = s) = q_s$ and $\Pr(M_{t+1} = D \mid M_t = s) = 1 - q_s$. Assume $M_0 = U$.

    i. [3 marks] Using the martingale (no-arbitrage) condition, find the risk-neutral transition probabilities q_U and q_D.Compute the risk-neutral transition matrix $P_{\mathbb{Q}}$ for the Markov chain $\{M_t\}$.

    ii. [2 mark] Using this model, price a two-period European call option with strike $K = 100$.

    (b) [2 marks] Prove that a harmonic function on an finite irreducible markov chain is constant. [Hint: Look at the maximum value of the function. Where have you used irreducibility?]

    (c) [3 marks] If S, T are stopping times, prove $S + T$ is a stopping time.

13. Answer the following questions:

    (a) [3 marks] Derive the joint distribution of the first two arrival times T_1, T_2 for a Poisson process of rate λ.

    (b) [4 marks] Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix $P = (p_{ij})$ on a countable state space S. For states $i, j \in S$, write $i \to j$ if there exists $n \geq 1$ such that $p_{ij}^{(n)} > 0$, and let

    $$f_{ij} = \mathbb{P}_i(T_j < \infty), \qquad T_j = \min\{n \geq 1 : X_n = j\}.$$

    Prove that if j is recurrent and $j \to i$, then $i \to j$ and $f_{ij} = 1$.

    (c) [3 marks] Prove the identity

    $$\mathbb{E}[XZ \mid \mathcal{G}] = Z\,\mathbb{E}[X \mid \mathcal{G}],$$

    for the case where $\mathcal{G} = \sigma(A_1, \ldots, A_m)$ is the σ-algebra generated by a finite partition $\{A_1, \ldots, A_m\}$ of Ω. You may assume that conditional expectations are represented as linear combinations of the indicator functions of the atoms A_i.

14. Answer the following questions:

    (a) [4 marks] Let $\{N(t),\, t \geq 0\}$ be a Poisson process with rate $\lambda > 0$, and let T_k denote the time of the k-th arrival. For a fixed $t > 0$, sketch the graph

    $$g(s) = \mathbb{E}[N(t) \mid T_2 = s]$$

    as a function of s.

(b) [3 marks] Let Z be a random variable. Consider the process $X_0 = 0$ and $X_n = X_{n-1} + Z$ for $n \geq 1$. **Prove or disprove**: The process X has stationary increments but does not have independent increments.

(c) [3 marks] Let $W(t)$ be a Wiener process. Find the conditional expectation of $W(3)$ given $W(6) = 1$.

15. Answer the following questions:

(a) [3 marks] Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with probabilities

$$P(\{\omega_1\}) = \tfrac{1}{2}, \quad P(\{\omega_2\}) = \tfrac{1}{4}, \quad P(\{\omega_3\}) = \tfrac{1}{8}, \quad P(\{\omega_4\}) = \tfrac{1}{8}.$$

Define two σ-algebras

$$\mathcal{F}_1 = \sigma(\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}), \qquad \mathcal{F}_2 = \sigma(\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}),$$

and the random variable $X = \mathbf{1}_{\{\omega_1\}}$. Let

$$Y_1 = E[X \mid \mathcal{F}_1], \qquad Y_2 = E[X \mid \mathcal{F}_2].$$

Compute

$$Z_1 := E[Y_1 \mid \mathcal{F}_2], \qquad Z_2 := E[Y_2 \mid \mathcal{F}_1],$$

and show that $Z_1 \neq Z_2$ as random variables.

(b) [4 marks] Let $(W_t)_{t \geq 0}$ be a standard Wiener process with $W_0 = 0$. Prove that the sample paths are almost surely not differentiable at 0.

(c) [3 marks] An institute offers a refund contract: pay now, and receive a refund $L =$ Rs. 100,000 in one year if no job is obtained. The stochastic discount factor is $M = M(S, m)$ where macro state $S \in \{\mathrm{R}, \mathrm{E}\}$ (recession/expansion) and marks $m \in \{\mathrm{H}, \mathrm{L}\}$ (high/low). The joint distribution and inputs are:

	$\mathbb{P}(S, m)$	$\mathbb{P}(\text{no job} \mid S, m)$	$M(S, m)$
(R, H)	0.12	0.30	1.05
(R, L)	0.28	0.60	1.00
(E, H)	0.36	0.10	0.88
(E, L)	0.24	0.40	0.91

Compute the fair price p of this financial instrument. Can you also compute the risk free rate of return?

16. An urn initially contains 1 black and 5 white balls. At each step $n \geq 1$, one ball is drawn uniformly at random, its color is recorded, and it is returned to the urn together with one additional ball of the same color. Let

$B_n =$ number of black balls after n draws, $\qquad W_n =$ number of white balls after n draws, $\qquad T_n = B_n + W_n$.

Define $\mathcal{F}_n = \sigma(B_0, W_0, \ldots, B_n, W_n)$ and $Z_n := B_n / T_n$.

(a) [3 marks] Show that $(B_n)_{n \geq 0}$ is a Markov process. Write its one-step transition probabilities at time n, and explain why it is not time-homogeneous.

(b) [2 marks] Show that $(Z_n)_{n \geq 0}$ is a martingale with respect to $(\mathcal{F}_n)$.

(c) [2 marks] Define

$$\tau = \inf\{n \geq 1 : \text{ the ball drawn at step } n \text{ is black}\}.$$

Assume that τ is a stopping time with respect to $(\mathcal{F}_n)$ and prove that

$$\Pr(\tau > m) \longrightarrow 0 \quad \text{as } m \to \infty.$$

Deduce that $\tau < \infty$ almost surely.

(d) [3 marks] Verify all hypotheses required to apply the Optional Stopping Theorem for the martingale Z_n, and hence prove

$$\mathbb{E}\left[\frac{1}{5+\tau}\right] = \frac{1}{12}.$$

17. Answer the following questions:

(a) A small farmer starts the season with one unit of grain. If she consumes it immediately, she receives utility 7 and the process ends. If she plants it, the harvest next period depends on the weather regime $W_t \in \{\mathrm{G}, \mathrm{B}\}$ (Good or Bad). The regime at $t = 1$ is observed before the decision is made and evolves as a two-state Markov chain $\{W_t\}$ with transition matrix

$$\begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix},$$

where rows correspond to the current state W_1 and columns to the next state W_2.

If she plants, then at $t = 2$ the utility from consuming the harvest is 12 if $W_2 = \mathrm{G}$ and 4 if $W_2 = \mathrm{B}$. There is no discounting between periods.

(a) [2 marks] Specify the state space, the action set, the transition rule, and the reward function.

(b) [2 marks] Write down the Bellman equations for periods 2 and 1, and compute their numerical values.

(c) [1 mark] State the farmer's optimal policy and the corresponding value function.

(b) [3 marks] Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls, and 2 blue balls; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage, a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to that urn. In the long run, what proportion of the selected balls are red? Just a numerical answer without proof fetches no marks. [Hint:Create a Markov Chain.]

(c) [2 marks] Let $(X_n)_{n\geq 0}$ be a martingale with respect to $(\mathcal{F}_n)$.

Show that its increments are pairwise orthogonal, i.e.

$$\mathbb{E}\big[(X_{n+1} - X_n)(X_{m+1} - X_m)\big] = 0 \quad \text{for all } n \neq m.$$