DE13: Week 8

1. In class we wrote the Kolmogorov axioms of probability theory. Recall that elements of sigma-algebra are called events. Using the axioms show the following:

   (a) If A is an event, then $\Pr(A) + \Pr(A^c) = 1$.

   (b) If $A \subseteq B$ are events, then $\Pr(A) \leq \Pr(B)$.

   (c) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$ are events, then

   $$\Pr(\cap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \Pr(A_n).$$

   [Hint: Freely use both axiom 3 or 3'. Draw Venn diagrams and guess the steps of the proof from your drawings. But every equality HAS to be justified by an axiom of Kolmogorov.]

2. Then we discussed an example where $\Omega = \mathbb{R}$ and the sigma-algebra $\mathcal{F}$ of events was generated by intervals of the type $(-\infty, b]$ where $b \in \mathbb{R}$. Then we defined a function: For any real number b, $F(b) := \Pr\{(-\infty, b]\}$. Prove the following three properties of F:

   (a) $\lim_{x \to \infty} F(x) = 1$,$\lim_{x \to -\infty} F(x) = 0$.

   (b) $F(x)$ is non-decreasing function of x. In other words, if $a > b$, then $F(a) \geq F(b)$.

   (c) $\lim_{x \to b+} F(x) = F(b)$. The converse is also true but it is much harder to prove. So you may just assume the converse for this course: If you have a function F that satisfies the above three properties, then the probability measure defined by $\Pr\{(-\infty, b]\} := F(b)$ satisfies Kolmogorov's axioms. Thus we generally call the above three properties as the defining properties of the cdf.

3. For all real x, let

4.

$$F(x) = \frac{1}{2} + \frac{x}{2\sqrt{2}\sqrt{1 + \frac{x^2}{2}}}.$$

Prove or disprove: F satisfies the three defining properties of the c.d.f.