DE13: Week 11

Continuous time Markov chains, SP, MSE

1. If

$$A = \begin{bmatrix} -a & b \\ a & -b \end{bmatrix},$$

then do the following

(a) Diagonalize the matrix. In other words, write the matrix as

$$A = S\Lambda S^{-1}.$$

Find S, Λ.

(b) Compute e^{At} using part (a).

2. Consider a continuous Markov chain with two states $S = \{0, 1\}$ with transition matrix for any $t \geq 0$ is given by

$$P(t) = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} \\ \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} \end{bmatrix}.$$

(a) Find the rate of transition matrix R.

(b) Show that $P'(t) = RP(t) = P(t)R$.

3. Let λ be a positive real number. For any whole number j, let $R_{j,j} = -\lambda$, $R_{j+1,j} = \lambda$ and suppose $R_{ij} = 0$ for any whole number $i \notin \{j, j+1\}$. Now consider a "system of coupled differential equations"

$$P'_{ij}(t) = \sum_{k=0}^{\infty} R_{ik} P_{kj}(t)$$

with initial conditions $P_{jj}(0) = 1$ and $P_{ij}(0) = 0$ for $i \neq j$.

(a) Expand the right hand side of the formula above. There should be two terms.

(b) Using the methods of ODE, derive the solutions for $P_{00}(t), P_{10}(t), P_{11}(t)$.

(c) Prove that an appropriate pmf value of Poisson distribution satisfies the general equation. Using the existence of uniqueness theorem for ODEs, conclude that this is the only solution.

4. Let the rate of transition matrix R of a two state Markov chain (with states $S = \{0, 1\}$) be

$$R = \begin{bmatrix} -\lambda & \mu \\ \lambda & -\mu \end{bmatrix}.$$

(a) Expanding $P'(t) = RP(t)$ and $P(0) = I$ write down the system of four coupled differential equations.

(b) **Method 1:** Solve the system of equations using techniques you learnt from ODE class. [Hint: columns of P(t) sum to 1, at all times t.]

(c) **Method 2:** Directly solve the system of equations by computing $P(t) = e^{Rt}$ and using problem 1.