Stochastic Process: First Internals

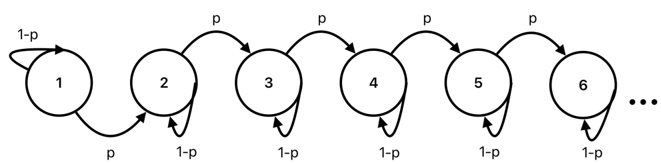
Guidelines

- The maximum marks that can be scored in this test is **40 marks** and you have **two hours**. You can use a calculator.
- If you score above 40, the additional marks will be added to the assigment total.
- You may use any theorem proved in class or any formula derived in class. Any other claim needs to be justified.

Problems

1. [6 marks] For the following problems, write down the final answer with one line of justification.

   (a) Consider a DTMC on the states of all natural numbers with $p = \frac{1}{7}$. The state graph is shown in Figure 1.

   What is the value of

   $$\lim_{n\to\infty} \Pr\{X_n = 2024\}?$$

   (b) $X_{n+1} = X_n + E_n$ for $n \geq 0$ with $X_0 = 0$. Suppose $\{E_n\}$ random variables are independent and identically distributed according to the Poisson distribution with mean 1, then compute $\Pr\{X_{100} = 37 \mid X_{98} = 35\}$.

   (c) Fill in the blanks: The states of an open communicating class are _______ while the states of a finite closed communicating class are ___________.

2. [3 marks] Define the following properties of a Markov chain:

   (a) Markov property of a stochastic process.

   (b) Time homogenity of a Markov chain.

   (c) Communicating class.

3. [2 marks] Write an example of a three state DTMC with only three communicating classes: two closed and one open. You can draw a state graph.

4. [2 marks]The matrix M is defined as

$$M = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

   Prove or disprove: For some natural number n,the matrix M^n has only positive entries.

5. [3 marks] Give an example of a process that does not have the independent increment property. You have to prove your claims.

6. [3 marks] Suppose there are three types of laundry detergent, 1, 2, and 3, and let X_n be the brand chosen on the nth purchase. Customers who try these brands are satisfied and choose the same thing again with probabilities 0.8, 0.6, and 0.4 respectively. When they change they pick one of the other two brands at random. Let P be the transition probability matrix for the markov chain X, then compute the limit

$$\lim_{n\to\infty} \frac{I + P + P^2 + \ldots + P^n}{n+1}.$$

7. [4 marks] Suppose a video game store sells game consoles and uses the following policy to restock consoles: If there are no consoles at the end of the day, the store orders enough to have 2 consoles ready the next day morning. If there are consoles at the end of the day, the store does not order new consoles. Suppose the distribution of demand $\{D_n\}$ for the number of consoles each day is known to be i.i.d process, where each random variable D_n has the p.m.f

$$f(0) = 0.5, f(1) = 0.3, f(2) = 0.2.$$

   Let I_n denote the number of consoles in the inventory at the end of day n. Clearly $I = \{I_n\}$ is a discrete time Markov chain. Further suppose the game store makes a profit of Rs.6000 per console sold but it costs Rs.1000 per day to run the store. We have been told that the game store opens with 2 consoles.

   (a) Write the t.p.m.

   (b) The owner becomes happy only when all the consoles are sold at the end of the day.What is the mean time that elapses between owner's happy states?

8. [4 marks] A Markov process moves on the integers 1, 2, 3, 4, and 5. It starts at 1 and, on each successive step, moves to an integer greater than its present position, moving with equal probability to each of the remaining larger integers. State five is an absorbing state.

   (a) Draw the state graph.

   (b) Find the expected number of steps to reach state five.

9. A discrete time Markov chain has the following transition probability matrix

$$P = \left[\quad \right]$$

Bonus Questions:

1. [4 marks] In the Leontief economic model, there are n industries $1, 2, \ldots, n$. The jth industry requires an amount $0 \leq q_{ij} \leq 1$ of goods (in rupee value) from company i to produce 1 rupee's worth of goods. The outside demand on the industries, in rupee value, is given by the vector $\mathbf{d}= (d_1, d_2, \ldots, d_n)$. Let $\mathbf{Q}$ be the matrix with entries q_{ij} .

   (a) In order to meet the outside demand $\mathbf{d}$ and the internal demands the industries must produce total amounts given by a vector $\mathbf{x}= (x_1, x_2, \ldots, x_n)$. Prove that

$$x = Qx + d.$$

   (b) Show that if Q is the Q-matrix block of a transition probability matrix of an absorbing markov chain, then it is possible to meet any outside demand $\mathbf{d}$.