Internals-II: Time Series Analysis
Instructor: Srikanth Pai, MSE, Chennai

Total Marks: 26 **Time:** 90 minutes

Instructions: Attempt all questions for max marks 20. All claims must be justified by derivation or reference to definitions and theorems. Freely use any theorem or calculations done in class but with reference.

1. (8 marks) Write a note on PACF and its use. Justify your points using theorems and computations.

2. (8 marks) Attempt exactly one of the two:

   (a) Monthly rainfall anomaly in Chennai (deviation in mm from the 30-year seasonal average) is modelled as an MA(1) process

   $$y_t = \varepsilon_t + 0.5\,\varepsilon_{t-1}, \qquad \varepsilon_t \sim \mathrm{WN}(0,1).$$

   The observed anomalies for the last two months are $y_{T-1} = 0.85$ and $y_T = 0.34$.

   i. (3 marks) Compute the optimal one-step-ahead forecast $\hat{y}_{T+1|T}$ using y_T and y_{T-1}.

   ii. (5 marks) The anomaly $y_{T+1} = 0.85$ is now observed. State the forecast update theorem precisely, identify each ingredient, and hence compute $\hat{y}_{T+2|T+1}$.

   (b) Let $\{Y_t\}$ be an ARMA(1,1) process

   $$(1-\phi L)Y_t = (1+\theta L)\varepsilon_t, \qquad |\phi|<1, \quad |\theta|<1, \quad \varepsilon_t \sim \mathrm{WN}(0,\sigma^2).$$

   Using the Wiener-Kolmogorov theorem, show that

   $$\hat{Y}_{t+1|t} = \phi Y_t + \theta \varepsilon_t.$$

3. (10 marks) Let $\{Y_t\}$ be an ARMA(1,1) process

   $$(1-\phi L)Y_t = (1+\theta L)\varepsilon_t, \qquad |\phi|<1, \quad \varepsilon_t \sim \mathrm{WN}(0,\sigma^2).$$

   The autocovariance generating function is defined as $g(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k$.

   (a) (2 marks) Show that

   $$g(z) = \sigma^2 \frac{(1+\theta^2) + \theta(z+z^{-1})}{(1+\phi^2) - \phi(z+z^{-1})}.$$

   (b) (3 marks) Derive the following three equations using the previous part:

   $$\begin{aligned} k=0: &\quad (1+\phi^2)\gamma_0 - 2\phi\gamma_1 = \sigma^2(1+\theta^2), \\ k=1: &\quad (1+\phi^2)\gamma_1 - \phi\gamma_0 - \phi\gamma_2 = \sigma^2\theta, \\ |k|\geq 2: &\quad (1+\phi^2)\gamma_k - \phi\gamma_{k-1} - \phi\gamma_{k+1} = 0. \end{aligned}$$

   (c) (3 marks) Solve the $|k| \geq 2$ recursion, by finding the characteristic roots. Discard the explosive root, and hence solve the $k = 0, 1$ system to obtain explicit expressions for γ_0 and γ_1.

   (d) (2 marks) Verify that $\gamma_1 = \phi\gamma_0 + \theta\sigma^2$ and that $\gamma_k = \phi^{k-1}\gamma_1$ for all $k \geq 1$.