# Internals-I: Time Series Analysis
# Instructor: Srikanth Pai

---

**Total Marks:** 32 **Time:** 105 minutes

**Instructions:** Attempt all questions. All claims must be justified by derivation or reference to definitions and theorems. Incomplete or unjustified answers will receive minimal credit.

1. (6 marks) Consider Samuelsons model:

$$c_t = \alpha y_{t-1}, \quad i_t = \beta(y_{t-1} - y_{t-2}), \quad y_t = c_t + i_t + g_t,$$

where $y_t$ is output, $c_t$ is consumption, $i_t$ is investment, $g_t$ is exogenous spending, and:

$$g_t = \varepsilon_t + \theta\varepsilon_{t-1}, \qquad \{\varepsilon_t\} \sim \text{WN}(0, \sigma^2).$$

Parameters satisfy $0 < \alpha < 1$ and $\beta > 0$.

(a) (4 marks) Substitute the consumption and investment equations into the accounting identity and derive a second-order difference equation for $y_t$. Express the result in lag-operator form:

$$\phi(L)y_t = \psi(L)\varepsilon_t.$$

The lag polynomials $\phi(L), \psi(L)$ must be explicit functions of parameters given above.

(b) (2 marks) Consider two specifications: (i) $\theta = 0$ (white noise government spending) and
(ii) $\theta = 0.8$ (moving-average government spending). How does the cycle length differ across these two examples?

2. (8 marks) Standard derivations

(a) (4 marks) Derive the autocovariance function $\gamma(k) = \text{Cov}(y_t, y_{t-k})$ for an MA(2) process from first principles.

(b) (4 marks) Now consider an AR(1) process:

$$y_t = \phi y_{t-1} + \varepsilon_t, \quad |\phi| < 1.$$

Derive the MA($\infty$) representation by solving for $y_t$ recursively. Justify that the series converges. Show that $\gamma(k) = \phi^{|k|}\gamma(0)$ for all $k \in \mathbb{Z}$, where $\gamma$ is the autocovariance function.

3. (10 marks) Consider the AR(3) process:

$$y_t = 0.5y_{t-1} - 0.2y_{t-2} + 0.1y_{t-3} + \varepsilon_t, \quad \varepsilon_t \sim \text{WN}(0, \sigma^2).$$

(a) (3 marks) Write out the Yule–Walker equations (a linear system) for $k = 0, 1, 2, 3$ explicitly for the given AR(3). [No derivation needed.]

(b) (7 marks) Show that the sample mean $\bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t$ satisfies:

$$\bar{y}_T \xrightarrow{p} 0.$$

Justify all steps using theorems proved in class. The theorems have to be stated precisely and then you should state how you are applying the theorem. [Warning: $y_t$ is not i.i.d!]

4. (8 marks) Consider two recurrences:

$$y_t = 2.3y_{t-1} - 1.2y_{t-2} + u_t - 1.5u_{t-1}$$

$$z_t = 0.8z_{t-1} + u_t$$

where $u_t = 0$ for $t \leq 0$ and $u_t = 1$ for $t \geq 1$.

With initial conditions $y_t = z_t = 0$ for all $t \leq 0$, show that the sequence $z$ is bounded. Finally, is

$$\sum_{t=0}^{\infty} |y_t - z_t|$$

finite? Only justified answers allowed.