Time Series Analysis - Homework 4

Srikanth Pai, Asst. Prof, MSE

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1. Let X and Y be random variables with finite second moments. Compute

$$\mathbb{E}(3X - 2Y + 5), \qquad \operatorname{Var}(3X - 2Y)$$

in terms of $\mathbb{E}X$, $\mathbb{E}Y$, $\operatorname{Var}(X)$, $\operatorname{Var}(Y)$, and $\operatorname{Cov}(X, Y)$.

2. Let (X, Y) be a bivariate random vector with

$$\mathbb{E}X = 1, \quad \mathbb{E}Y = 2, \quad \operatorname{Var}(X) = 1, \quad \operatorname{Var}(Y) = 4, \quad \operatorname{Cov}(X, Y) = -1.$$

Compute the mean and variance of $X + 2Y$.

3. Let X be a random variable with $\mathbb{E}X = 0$ and $\operatorname{Var}(X) = \sigma^2$. Let $Y = X + \varepsilon$, where ε is independent of X with mean zero and variance τ^2.

Compute $\mathbb{E}(Y \mid X)$ and $\operatorname{Var}(Y \mid X)$.

4. Suppose observed data satisfy

$$Y_i = X_i + \frac{1}{\sqrt{n}} u_i,$$

where u_i has mean zero and finite variance and is independent of X_i. Let $\bar{Y}_n$ be the sample mean of Y_i. Describe the large-sample behavior of $\bar{Y}_n$ and interpret the role of the scaling $1/\sqrt{n}$.

5. Let

$$\hat{\theta}_n = \frac{\bar{X}_n}{\bar{Y}_n},$$

where $\bar{X}_n$ and $\bar{Y}_n$ are sample means and $\mathbb{E}Y_i \neq 0$. Determine the limit of $\hat{\theta}_n$ as the sample size increases, and explain why such ratio estimators arise naturally in applications.

6. An estimator satisfies

$$\hat{\theta}_n = \theta_0 + \frac{1}{\sqrt{n}} Z_n,$$

where Z_n converges in distribution to a nondegenerate random variable. Explain why $\hat{\theta}_n$ is close to θ_0 for large samples, and why rescaling by $\sqrt{n}$ is necessary to study its distribution.

7. Let $\{X_t\}_{t \geq 1}$ be a sequence of random variables with

$$\mathbb{E}[X_t] = \mu, \qquad \operatorname{Var}(X_t) = \sigma^2 < \infty.$$

Suppose that the following central limit theorem holds:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} (X_t - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

Define the sample mean

$$\bar{X}_T = \frac{1}{T}\sum_{t=1}^{T} X_t.$$

(a) Show that $\bar{X}_T \xrightarrow{p} \mu$.

(b) Find the asymptotic distribution of

$$\sqrt{T}(\bar{X}_T - \mu).$$

Explain briefly how this result is used for large-sample inference.

8. Prove that uncorrelated jointly normal random variables are independent. [Hint:Use characteristic function!]

9. Show that if X, Y is bivariate normal, then $\mathbb{E}[X|Y]$ is also normally distributed. Compute the mean and variance of this random variable. [Hint: Consider a regression of X vs Y.]

10. Work out the pdf of bivariate normal explicitly starting from the matrix form in terms of means, variances and correlation coefficient of the pair of random variables.