Time Series Analysis - Homework 2

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Recall

We considered the scalar difference equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t,$$

where $\{w_t\}$ is a deterministic sequence. Defining the $p \times 1$ state vector

$$\xi_t = (y_t, y_{t-1}, \ldots, y_{t-p+1})',$$

the system can be written in first-order vector form as

$$\xi_t = F\xi_{t-1} + v_t, \qquad v_t = (w_t, 0, \ldots, 0)'.$$

Given ξ_{-1}, repeated substitution yields

$$\xi_{t+j} = F^{j+1}\xi_{-1} + \sum_{k=0}^{j} F^k v_{t+j-k}.$$

The scalar variable is recovered as

$$y_{t+j} = e_1'\xi_{t+j}, \qquad e_1 = (1, 0, \ldots, 0)'.$$

Defining the dynamic multiplier

$$\psi_j = \frac{\partial y_{t+j}}{\partial w_t},$$

we showed that

$$\psi_j = e_1' F^j e_1,$$

which depends only on F and j, and not on calendar time.

Homework

1. Inflation is assumed to adjust gradually over time, with possible overshooting. Consider the second-order difference equation

   $$\pi_t = (1 + \gamma)\pi_{t-1} - \gamma\,\pi_{t-2} - \kappa u_t,$$

   where π_t denotes inflation and u_t denotes unemployment. Unemployment is treated as exogenous.

   The following code simulates the inflation path generated by this equation.

```python
import numpy as np
import matplotlib.pyplot as plt

T = 60

gamma = 0.6      # adjustment parameter
kappa = 0.5      # unemployment sensitivity

u = np.zeros(T)
u[0] = 1.0       # one-time unemployment shock

pi = np.zeros(T)
pi[0] = 0.0
pi[1] = -kappa*u[0]

for t in range(2, T):
    pi[t] = (1 + gamma)*pi[t-1] - gamma*pi[t-2]

plt.plot(pi)
plt.axhline(0, linestyle="--")
plt.xlabel("t")
plt.ylabel("Inflation")
plt.show()
```

(a) Run the code as written and describe the time path of inflation.

(b) (Experiment) Modify the value of γ and experiment with different parameter values. Identify values of γ for which inflation exhibits oscillatory behavior.

(c) (Analytic/Mathematical) Explain your observations by analyzing the characteristic equation associated with the difference equation.

2. Assume F has p distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

   (a) For each eigenvalue λ_i, we construct an eigenvector q_i of F normalized so that its pth coordinate equals 1. Show that each such eigenvector must take the form

$$q_i = (\lambda_i^{p-1}, \lambda_i^{p-2}, \ldots, \lambda_i, 1)'.$$

   (b) Construct a $p \times p$ matrix by stacking the eigenvectors

$$T = (q_1, \ldots, q_p).$$

   Show that T is invertible by computing its determinant explicitly and using the assumption that the λ_i are distinct. [Hint: Freely use results about Vandermonde matrices for now. You do not have to prove any of it.]

3. (a) Show that

$$F = T\Lambda T^{-1}, \qquad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p).$$

(b) Deduce that

$$F^j = T\Lambda^j T^{-1} \quad \text{for all } j \geq 0.$$

4. (a) Using $\psi_j = e_1' F^j e_1$, show that there exist constants $c_1, \ldots, c_p$ such that

$$\psi_j = \sum_{i=1}^{p} c_i \lambda_i^j \quad \text{for all } j \geq 0.$$

(b) Using $\psi_0 = 1$, prove that

$$\sum_{i=1}^{p} c_i = 1.$$

(c) Prove that

$$c_i = \frac{\lambda_i^{p-1}}{\prod_{k \neq i}(\lambda_i - \lambda_k)}.$$

5. (Samuelson multiplier–accelerator model)

This problem illustrates how output may adjust in an oscillatory way following a change in government expenditure, due to delayed consumption and investment responses.

Consider a closed economy with income y_t, consumption c_t, and investment i_t given by

$$c_t = \alpha y_{t-1}, \qquad 0 < \alpha < 1,$$

$$i_t = \beta(y_{t-1} - y_{t-2}), \qquad \beta > 0,$$

and goods-market equilibrium

$$y_t = c_t + i_t + g_t,$$

where g_t is exogenous government expenditure.

(a) Show that y_t satisfies a second-order difference equation of the form

$$y_t = (\alpha + \beta)y_{t-1} - \beta y_{t-2} + g_t.$$

(b) Write the system in state-space form

$$\xi_t = F\xi_{t-1} + v_t,$$

using the same state vector convention as before.

(c) Compute the eigenvalues of F and show that they are complex if and only if

$$(\alpha + \beta)^2 < 4\beta.$$

(d) Suppose $g_t = \bar{g}$ for all $t \geq 0$. Show that deviations from steady state satisfy

$$\Delta y_{t+j} = c_1 \lambda_1^j + c_2 \lambda_2^j,$$

where λ_1, λ_2 are the eigenvalues of F.

(e) Show that if the eigenvalues are complex with modulus less than one, the economy exhibits damped oscillations in response to a permanent increase in g_t.

(f) Interpret α and β as the marginal propensity to consume and the accelerator parameter, and explain how their magnitudes affect the persistence and amplitude of cycles.

In the Samuelson multiplier–accelerator model, consumption depends on past income and investment depends on changes in income, leading to a second-order difference equation for output. For certain values of the marginal propensity to consume and the accelerator parameter, the associated transition matrix has complex eigenvalues, so the impulse response to a temporary government expenditure shock is oscillatory. These oscillations arise from the internal adjustment mechanism of the model rather than from persistence in the shock itself.