MADRAS SCHOOL OF ECONOMICS

UNDERGRADUATE PROGRAMME IN ECONOMICS (HONOURS) [2023-26]

SEMESTER 6 [JANUARY – APRIL, 2026]

REGULAR EXAMINATION, APRIL-MAY 2026

Course Name: Introduction to Time Series Analysis **Course Code:** DE09

Duration: 2 Hours **Maximum Marks: 60**

Instructions: *For part A write short answers (most preferably a single word,/single number/single sentence). For part B, writing relevant formulae and quoting correct definitions helps me give you part marks. For questions with qualitative answers, creative writing is prohibited.*

Part A: Answer all questions in one line. (1 mark x 10 questions = 10 marks)

Notations are as used in class.

1. What is the full form of PACF?
2. Define an integrated process of order n, written as $I(n)$.
3. State one implication of asymptotic normality of MLE for inference.
4. Write down the formula for Akaike Information Criteria. Briefly mention the meaning of the terms.
5. For a first order deterministic sequences $y_t = 0.3y_{t-1} + \epsilon_t$, write the impulse response function ψ_j as a function of j.
6. If $a_t + b_t = c_t$ and all three are $I(1)$, give a cointegrating vector for $(a, b, c)^T$.
7. What is the null hypothesis of the Ljung–Box test? You can describe it in words.
8. For $Y_t = 0.5Y_{t-1} + \epsilon_t$, what is $E[Y_{t+1} \mid \mathcal{F}_t]$?
9. Let $W(t)$ be a Wiener process. Compute $\mathbb{E}\left(\int_0^1 W(t)^2\, dt\right)$.
10. The process $Y_t = 1.5Y_{t-1} - 0.5Y_{t-2} + \epsilon_t$ is an ARIMA(p, d, q) process. What are the values of p, d, q?

Part B: Answer any five. (5 mark x 10 questions = 50 marks)

11. [10 marks] What is Granger causality? Explain how VAR models are used in detecting Granger causality. Illustrate the use with a real example.

12. [10 marks] State and prove the Wiener-Kolmogorov theorem. You may assume that the optimal MMSE forecast is conditional expectation.

13. Suppose a covariance-stationary process has sample autocorrelations $\hat{\rho}_1 = 0.6$ and $\hat{\rho}_2 = 0.2$.

    (a) [4 marks] Assuming the process is an AR(2) model, write down the Yule-Walker equations in matrix form.

    (b) [4 marks] Solve the system to obtain the method-of-moments estimators $\hat{\phi}_1$ and $\hat{\phi}_2$.

    (c) [2 marks] Verify that the estimated AR(2) process is covariance stationary.

14. A zero-mean covariance-stationary process $\{Y_t\}$ has autocovariances

$$\gamma_0 = 4, \qquad \gamma_1 = 2, \qquad \gamma_2 = 1.$$

Consider the one-step-ahead linear forecast using two lags:

$$\hat{Y}_{t+1|t} = \alpha_1 Y_t + \alpha_2 Y_{t-1}.$$

    (b) [6 marks] Find the optimal forecast coefficients α_1 and α_2.

    (c) [4 marks] Compute the mean squared forecast error and hence obtain the coefficient of determination $R^2 = 1 - \text{MSE}/\gamma_0$.

15. Consider the time series {2, 2, 3, 1, 2} and the AR(1) model

$$X_t = \phi X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2).$$

    (a) [4 marks] Conditioning on X_1, the conditional log-likelihood is given by

$$\ell(\phi, \sigma^2) = -\frac{T-1}{2}\log(2\pi) - \frac{T-1}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=2}^{T}(X_t - \phi X_{t-1})^2.$$

    Using the above expression, Prove that the maximum likelihood estimators are

$$\hat{\phi} = \frac{\sum_{t=2}^{T} X_t X_{t-1}}{\sum_{t=2}^{T} X_{t-1}^2}, \qquad \hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=2}^{T}(X_t - \hat{\phi}\, X_{t-1})^2,$$

    and hence show that the maximised log-likelihood reduces to

$$\ell(\hat{\phi}, \hat{\sigma}^2) = -\frac{T-1}{2}\log(\hat{\sigma}^2) - \frac{T-1}{2}\big(1 + \log 2\pi\big).$$

    (b) [3 marks] Compute $\hat{\phi}$ and $\hat{\sigma}^2$ for the given data.

    (c) [3 marks] Prove that for this model the AIC can be written as

$$\text{AIC} = 4\log\left(\frac{11\pi}{4}\right) + 8,$$

    and evaluate this expression numerically.

16. We know that that the autocovariance of a MA(1) process vanishes above lag 1. In this problem, I will help you prove the converse for invertible covariance stationary processes. Suppose (X_t) is invertible covariance stationary with autocovariance $\gamma(h) = 0$ for all $|h| \geq 2$.

   (a) [4 marks] Show that

   $$g_X(e^{-i\omega}) = \gamma(0) + 2\gamma(1)\cos\omega.$$

   Hint: Use $e^{i\omega} = \cos\omega + i\sin\omega$.

   (b) [4 marks] Equating the above expression to autocovariance generating function of $MA(1)$

   $$g_X(e^{-i\omega}) = \sigma^2 \left|1 + \theta e^{-i\omega}\right|^2$$

   derive a **unique** value for σ^2, θ.

   (c) [2 marks] Assuming the uniqueness of Fourier series representation, conclude that (X_t) is MA(1) process.

17. We will work out the spurious regression between two independent random walks. Consider two independent I(1) processes:

   $$y_t = y_{t-1} + u_t, \qquad x_t = x_{t-1} + v_t, \qquad t = 1, \ldots, T,$$

   where $u_t \overset{\text{iid}}{\sim} (0, \sigma^2)$, $v_t \overset{\text{iid}}{\sim} (0, \sigma^2)$, $\{u_t\}$ and $\{v_t\}$ are mutually independent, and $y_0 = x_0 = 0$. Consider the OLS regression *without intercept*: $y_t = \beta x_t + \varepsilon_t$, with coefficient of determination:

   $$R^2 = \frac{\left(\sum_{t=1}^T y_t x_t\right)^2}{\sum_{t=1}^T y_t^2 \cdot \sum_{t=1}^T x_t^2}.$$

   (a) [3 marks] By the functional central limit theorem (FCLT):

   $$\frac{1}{\sigma\sqrt{T}} y_{\lfloor rT \rfloor} \xrightarrow{d} W_1(r), \qquad \frac{1}{\sigma\sqrt{T}} x_{\lfloor rT \rfloor} \xrightarrow{d} W_2(r),$$

   where W_1, W_2 are independent standard Brownian motions on $[0, 1]$. Using the Riemann sum approximation, show that:

   $$\frac{1}{T^2} \sum_{t=1}^T y_t^2 \xrightarrow{d} \sigma^2 \int_0^1 W_1(r)^2 \, dr.$$

   (b) [4 marks] Using the continuous mapping theorem, show that:

   $$R^2 \xrightarrow{d} \frac{\left(\int_0^1 W_1(r)\, W_2(r)\, dr\right)^2}{\int_0^1 W_1(r)^2 \, dr \cdot \int_0^1 W_2(r)^2 \, dr}.$$

   (c) [3 marks] It turns out that the limiting distribution has $E[R^2] \approx 0.44$. Explain why this invalidates R^2 as a measure of fit for I(1) regressions, and state one remedy.