DE09:Time series analysis: Internals-2

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1st April 2025

Rules of the exam

1. 90 minutes, 22 marks paper. Maximum score is 20.
2. Anyone in posession of mobile/smartwatch or any electronic device capable of connecting to a network will be given a zero.
3. You are allowed to use definitions, theorems and formulae proved in class. Any other methods will not score marks allowed unless you prove all the relevant claims implicit in your method.
4. All the questions follow notations used in class. You may use a calculator.

Problems

1. [4 marks] Explain how to use gradient descent to estimate parameters of a MA(q) process.
2. [4 marks] Derive the infinite sample forecast for an AR(2) process using the Wiener-Kolmogorov representation.
3. [3 marks] The matrix
$$\begin{bmatrix} 8 & 4 \\ 4 & 10 \end{bmatrix}$$
can be written in the form LDL^T where L is lower triangular with unit diagonals and D is a diagonal matrix.
   (a) Compute L, D.
   (b) Is the matrix positive definite? Justify.
4. [3 marks] Chatgpt input: *Derive the AR(1) linear MMSE forecast that depends on the latest two samples*
   Chatgpt response: The Minimum Mean Squared Error (MMSE) forecast for an AR(1) process that depends on the latest two samples requires us to find the best linear predictor of X_{t+1} given X_t and X_{t-1}.
   $X_t = \phi X_{t-1} + \varepsilon_t$ (AR(1) process)
   $X_{t+1} = \phi X_t + \varepsilon_{t+1}$ (One-step ahead)
   The forecast is a linear combination:
   $\hat{X}_{t+1|t,t-1} = aX_t + bX_{t-1}$ (Linear predictor)
   Minimize MSE:
   $E[(X_{t+1} - aX_t - bX_{t-1})^2]$ (MSE minimization)
   Using second-order moments and solving normal equations:
$$a = \frac{\phi(1-\phi^2)}{1-\phi^2} \quad \text{(Optimal coefficient for } X_t\text{)}$$
$$b = \frac{\phi^2(1-\phi)}{1-\phi^2} \quad \text{(Optimal coefficient for } X_{t-1}\text{)}$$

Final MMSE forecast:

$$\hat{X}_{t+1|t,t-1} = \frac{\phi(1-\phi^2)}{1-\phi^2} X_t + \frac{\phi^2(1-\phi)}{1-\phi^2} X_{t-1}.$$

(a) [2 marks] Work out the correct forecast.

(b) [1 mark] Locate an error in the chatgpt argument.

5. [3 marks] We know that if the data for multilinear regression is written in the form

$$\mathbf{y} = \mathbf{X}\beta + \epsilon,$$

then the MMSE estimator (assuming ϵ is white noise) is of the form

$$\hat{\beta} = (\mathbf{X^T X})^{-1}\mathbf{X^T y},$$

Suppose we have the time series $y_1 = 1, y_2 = 2, y_3 = 3, y_4 = 4$ that follows AR(2) model. Assuming there is no intercept term in the AR(2) model, estimate the parameters of an AR(2) process using the formula given.[Hint: Write $\mathbf{X}$ as a 2×2 matrix.]

6. [5 marks] Let $Y_t = 1+0.5Y_{t-1}+\epsilon_t$ where ϵ_t is white noise with variance 0.25 units. Compute the autocorrelation function and partial autocorrelation function of this process.