Assignment 5: AIC,DF Test, and Box-Jenkins Framework

DE09 Time series analysis, Srikanth Pai, MSE

Essay questions

1. Define Kullback-Liebler divergence. Sketch Akaike's motivation for AIC as a model selection tool using the KL divergence. Explain it with the relevant equations.
2. Write a note on Dickey-Fuller test. Explain its working principle as a unit root test. [This involves explaining the hypothesis testing setup, the DF statistic, and its distribution.]
3. With the help of a flowchart explain all the steps of Box-Jenkins methodology. Dont forget to write down the relevant equations in your explanations. Justify assumptions using mathematical theorems (like Wold's theorem for example).

Maths practice

1. From integral calculus, we know that

$$\int_0^1 f(t)dt = \lim_{T\to\infty} \sum_{k=1}^{T} f\left(\frac{k}{T}\right)\frac{1}{T}. \qquad \text{(Riemann summation)}$$

   Solve the following problems using the above formula:

   (a) Does the integral value on LHS change if we start the summation on the right from $k = 0$?

   (b) $\lim_{T\to\infty} \frac{1}{T^3} \sum_{k=1}^{T} k^2$

   (c) If $\lim_{T\to\infty} \frac{1}{T^{10}} \sum_{k=1}^{T} k^n = \frac{1}{10}$ then what is the value of n?

   (d) Write the answer in terms of integrals of functions of $f(t)$:

$$\lim_{T\to\infty} \frac{1}{T^2} \sum_{k=1}^{T} kf\left(\frac{k}{T}\right).$$

Theory problems

1. If $y_t = \rho y_{t-1} + u_t$ where $\{u_t\}$ is white noise with mean σ^2 and $y_0 = 0$.

   (a) Show that the OLS estimate of $\rho$is

$$\widehat{\rho}_T = \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2}.$$

(b) Using central limit theorem, it is known that the statistic $\sqrt{T}(\widehat{\rho}_T - \rho)$ has a limiting distribution of $\mathcal{N}(0, 1-\rho^2)$.You may assume this. What happens happens to the limiting distribution as ρ approaches one? What is the limit of the limiting distribution?

(c) Rewrite

$$T(\widehat{\rho}_T - 1) = \frac{\frac{1}{T}\sum_{t=2}^{T} u_t y_{t-1}}{\frac{1}{T^2}\sum_{t=2}^{T} y_{t-1}^2}$$

Assuming $\rho = 1$, show that as $T \to \infty$, the distribution of the numerator tends to $\frac{\sigma^2}{2}(X-1)$ where $X \sim \chi^2(1)$. As $T \to \infty$, the expectation of the denominator tends to the constant $\frac{\sigma^2}{2}$.

2. Recall that the Wiener processes $W(t)$ are scaled limit of random walks. Let $\{u_t\}$ is white noise with mean σ^2. You may assume Hamilton's equation 17.3.8 that we have worked out in class.Now derive 17.3.22,17.3.26 and hence conclude rigorously that the limiting distribution of statistic in last part of previous problem is the distribution of

$$\frac{1/2(W(1)^2 - 1)}{\int_0^1 [W(r)]^2 dr}$$

Argue that 68% of the time the asymptotic statistic $\lim_{T\to\infty} T(\widehat{\rho}_T - 1)$ is negative.

3. Recall that the t-statistic of the slope in the OLS problem is

$$t_T = \frac{(\widehat{\rho}_T - 1)}{\text{SE}(\widehat{\rho})} = \frac{(\widehat{\rho}_T - 1)}{\left\{s_T^2 \div \sum_{t=2}^{T} y_{t-1}^2\right\}^{\frac{1}{2}}}$$

where

$$s_T^2 = \sum_{t=1}^{T} \frac{(y_t - \widehat{\rho}_T y_{t-1})^2}{T-1}$$

(a) (Optional for finals) Prove $\lim_{T\to\infty} s_T^2 = \sigma^2$.

(b) Conclude using part (a) that

$$\lim_{T\to\infty} t_T = \frac{1/2(W(1)^2 - 1)}{\sqrt{\int_0^1 [W(r)]^2 dr}}.$$

4. In the above theory (and in class), our derivations of DF test dont work when the time series has non-zero mean. Explain? [See the derivation for this case in Hamilton 17.4.36. Derivation is not testable for final exam.]

5. Show that the AIC formula for a model with normally distributed statistic (say regression residuals, or innovations of an MA process) can be chosen as

$$n \log \widehat{\sigma^2} + 2k$$

where $\widehat{\sigma^2}$ is the sample variance. [Hint: Although the AIC formula is slightly different, show that the above formula can be used following Akaike's idea.]

Numerical Problems

1. The AR(1) process for the nominal three-month U.S. Treasury bill rate was fitted by OLS regression to quarterly data, second quarter of 1947 to first quarter of 1989:

$$i_t = \rho i_{t-1}$$

The estimated value of ρ is 0.99694 with a standard error of 0.010592. Use both statistics in Theory problems 1.c and 3, along with Appendix Table B.5 and B.6 for finite sample DF critical values and decide whether the process is stationary for 5% confidence level.[This is worked out example in Hamilton.]

2. (Hand computation of Box-Jenkins) You are given a short, mean-zero time series:

$$Y = \{2,\ 1,\ -1,\ -2,\ -1\}$$

Answer the following:

(a) Compute the sample autocorrelations at lags 1 and 2.

(b) Based on your calculations, suggest whether an AR(1) or an MA(1) model might be more appropriate. Justify your choice briefly.

(c) Suppose you choose an AR(1) model. Estimate the model parameter and compute the residuals at each time point. Find the total squared residual error.

(d) Perform a Dickey-Fuller test to check whether the AR(1) model is stationary. Regress ΔY_t on Y_{t-1}, compute the test statistic, and comment on whether the unit root hypothesis seems plausible.

(e) Now fit an MA(1) model with $\theta = 0.5$. Estimate the residuals recursively and compute the total squared residual error for this model.

(f) Compute the Akaike Information Criterion (AIC) for both models using your results above. Which model is preferred based on AIC? [Hint: Use theory problem 5.]

(g) Using the model you prefer, forecast the next value Y_6 based on all the previous values.