Internal 1 Practice Tutorial

Lectures 10–12: Joint Distributions, Expectation, and Moment Generating Functions

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Textbook: Larsen & Marx, *An Introduction to Mathematical Statistics and Its Applications*, Fifth Edition.

Topics: Sections 3.7 and 3.11 (joint, marginal, conditional); the law of iterated expectation; moment generating functions; expectation via LOTUS; transformations of continuous random variables.

Part A. Conditional Distributions (§3.11)

1. Suppose X and Y have the joint pmf

$$p_{X,Y}(x,y) = \frac{x+y+xy}{21}$$

for the points $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, where X denotes a "message" sent (either $x=1$ or $x=2$) and Y denotes a "message" received. Find the probability that the message sent was the message received, that is, find $p_{Y|x}(y)$.

2. An urn contains eight red chips, six white chips, and four blue chips. A sample of size 3 is drawn without replacement. Let X denote the number of red chips in the sample and Y, the number of white chips. Find an expression for $p_{Y|x}(y)$.

3. Let X denote the number on a chip drawn at random from an urn containing three chips, numbered 1, 2, and 3. Let Y be the number of heads that occur when a fair coin is tossed X times. Compute $P(X=2 \mid Y=2)$.

4. Find the conditional pdf of Y given x if

$$f_{X,Y}(x,y) = x+y, \quad 0 \le x \le 1,\ 0 \le y \le 1.$$

5. Suppose that X and Y are distributed according to the joint pdf

$$f_{X,Y}(x,y) = \frac{2}{5}(2x+3y), \quad 0 \le x \le 1,\ 0 \le y \le 1.$$

Find

(a) $f_X(x)$.
(b) $f_{Y|x}(y)$.
(c) $P\left(\frac{1}{2} \le Y \le \frac{3}{4} \,\middle|\, X = \frac{1}{2}\right)$.
(d) $E(Y \mid x)$.

Part B. The Law of Iterated Expectation

6. Let X be the number on a chip drawn at random from an urn containing three chips, numbered 1, 2, and 3. Given $X = x$, a fair coin is tossed x times; let Y be the number of heads.

   (a) Find $E(Y \mid X = x)$, and hence write the random variable $E(Y \mid X)$ as a function of X.

   (b) Apply the law of iterated expectation, $E(Y) = E\big[E(Y \mid X)\big]$, to find $E(Y)$.

7. Let X be uniform on $(0, 1)$. Given $X = x$, suppose Y is uniform on $(x, 1)$.

   (a) Explain why $E(Y \mid X)$ is a random variable, and find it as a function of X.

   (b) Use the law of iterated expectation to find $E(Y)$.

Part C. Practice Problems

8. If $p_{X,Y}(x, y) = cxy$ at the points $(1, 1)$, $(2, 1)$, $(2, 2)$, and $(3, 1)$, and equals 0 elsewhere, find c.

9. Suppose that random variables X and Y vary in accordance with the joint pdf $f_{X,Y}(x, y) = c(x + y)$, $0 < x < y < 1$. Find c.

10. Suppose that X and Y have a bivariate uniform density over the unit square:

$$f_{X,Y}(x, y) = c, \quad 0 < x < 1,\ 0 < y < 1.$$

   (a) Find c.

   (b) Find $P\big(0 < X < \frac{1}{2},\ 0 < Y < \frac{1}{4}\big)$.

11. Are the random variables X and Y independent if $f_{X,Y}(x, y) = \frac{2}{3}(x+2y)$, $0 \le x \le 1$, $0 \le y \le 1$? Justify your answer.

12. Suppose that $f_{X,Y}(x, y) = 6(1 - x - y)$ for x and y satisfying $0 \le x + y \le 1$, $x \ge 0$, $y \ge 0$.

   (a) Sketch the support region.

   (b) Find the marginal pdf $f_X(x)$.

13. Two fair dice are tossed independently. Let X denote the number appearing on the first die and Y the number on the second.

   (a) Write down $p_{X,Y}(x, y)$.

   (b) Show that X and Y are independent.

14. A die is rolled six times. Let X be the total number of 4's that occur and let Y be the number of 4's in the first two tosses. Find $p_{Y|x}(y)$.

15. Five cards are dealt from a standard poker deck. Let X be the number of aces received and Y the number of kings. Compute $P(X = 2 \mid Y = 2)$.

16. Suppose that

$$f_{Y|x}(y) = \frac{2y + 4x}{1 + 4x} \quad \text{and} \quad f_X(x) = \tfrac{1}{3}(1 + 4x), \quad 0 < x < 1,\ 0 < y < 1.$$

   Find the marginal pdf $f_Y(y)$.

17. Let X have the geometric pmf

$$p_X(k) = (1-p)^{k-1}\, p, \qquad k = 1, 2, 3, \ldots, \quad 0 < p < 1.$$

   (a) Show that

$$M_X(t) = \frac{p\, e^t}{1 - (1-p)e^t}, \qquad e^t < \frac{1}{1-p}.$$

   (b) Use $M_X(t)$ to compute $E(X^2)$.

18. Let X have pdf $f_X(x) = \lambda e^{-\lambda x}$, $x > 0$, $\lambda > 0$. Find $M_X(t)$ for $t < \lambda$, and use it to obtain $E(X)$ and $E(X^2)$.

19. Let X have pdf $f_X(x) = 3x^2$, $0 < x < 1$. Using LOTUS, compute $E(X^2)$ and $E\left(\dfrac{1}{X}\right)$.

20. Let X be uniform on $(-1, 1)$, so $f_X(x) = \frac{1}{2}$, $-1 < x < 1$. Using LOTUS, compute $E(X^2)$ and $E(|X|)$.

21. Let X be uniform on $(0, 1)$. Let $Y = -\ln X$. Find the pdf of Y and state its support.

22. Let X have pdf $f_X(x) = 2x$, $0 < x < 1$. Let $Y = X^2$. Find the pdf of Y and state its support.

23. Let X have pdf

$$f_X(x) = c\, x(1-x), \qquad 0 < x < 1,$$

   and $f_X(x) = 0$ otherwise.

   (a) For what value of c is f_X a valid pdf?
   (b) Find $E(X)$.
   (c) Find $\mathrm{Var}(X)$.

Answer Key — Part A

1. The conditional pmfs are:

$$p_{Y|1}(1) = \tfrac{3}{8}, \quad p_{Y|1}(2) = \tfrac{5}{8}; \qquad p_{Y|2}(1) = \tfrac{5}{13}, \quad p_{Y|2}(2) = \tfrac{8}{13}.$$

   The probability that the message received equals the message sent is $P(Y = X) = \frac{11}{21}$.

2. $p_{Y|x}(y) = \dfrac{\dbinom{6}{y}\dbinom{4}{3-x-y}}{\dbinom{10}{3-x}}, \qquad y = 0, 1, \ldots, \min(3-x,\, 6).$

3. $P(X = 2 \mid Y = 2) = \dfrac{3}{4}.$

4. $f_{Y|x}(y) = \dfrac{x+y}{x+\frac{1}{2}} = \dfrac{2(x+y)}{2x+1}, \qquad 0 \le y \le 1.$

5. (a) $f_X(x) = \dfrac{4x+3}{5}, \quad 0 \le x \le 1.$

   (b) $f_{Y|x}(y) = \dfrac{2(2x+3y)}{4x+3}, \quad 0 \le y \le 1.$

   (c) $P\big(\frac{1}{2} \le Y \le \frac{3}{4} \mid X = \frac{1}{2}\big) = \dfrac{23}{80}.$

   (d) $E(Y \mid x) = \dfrac{2(x+1)}{4x+3}.$

Answer Key — Part B

6. (a) Given $X = x$, Y is binomial$(x, \frac{1}{2})$, so $E(Y \mid X = x) = \dfrac{x}{2}$. Hence $E(Y \mid X) = \dfrac{X}{2}$.

   (b) With X uniform on $\{1, 2, 3\}$, $E(X) = 2$, so $E(Y) = E\left(\dfrac{X}{2}\right) = \dfrac{1}{2}\,E(X) = 1.$

7. (a) $E(Y \mid X = x) = \dfrac{x+1}{2}$. Substituting the random variable X gives $E(Y \mid X) = \dfrac{X+1}{2}$, a function of X; since X is random, so is $E(Y \mid X)$.

   (b) $E(Y) = E\left(\dfrac{X+1}{2}\right) = \dfrac{E(X)+1}{2} = \dfrac{\frac{1}{2}+1}{2} = \dfrac{3}{4}.$

Answer Key — Part C

8. $c = \dfrac{1}{10}.$

9. $c = 2.$

10. (a) $c = 1.$ (b) $P = \dfrac{1}{8}.$

11. **Not independent.**

12. (a) Triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$.

    (b) $f_X(x) = 3(1-x)^2, \quad 0 \le x \le 1.$

13. (a) $p_{X,Y}(x,y) = \dfrac{1}{36}$ for $x, y \in \{1, 2, 3, 4, 5, 6\}$.

    (b) $p_{X,Y}(x,y) = \dfrac{1}{36} = \dfrac{1}{6} \cdot \dfrac{1}{6} = p_X(x)\,p_Y(y)$ for all x, y. **Independent.**

14. $p_{Y|x}(y) = \binom{2}{y}\big(\frac{1}{6}\big)^y\big(\frac{5}{6}\big)^{2-y} \cdot \binom{4}{x-y}\big(\frac{1}{6}\big)^{x-y}\big(\frac{5}{6}\big)^{4-(x-y)} \Big/ \binom{6}{x}\big(\frac{1}{6}\big)^x\big(\frac{5}{6}\big)^{6-x}$, which simplifies to the hypergeometric probability $\dfrac{\binom{2}{y}\binom{4}{x-y}}{\binom{6}{x}}, \quad y = 0, 1, \ldots, \min(2, x).$

15. $P(X = 2 \mid Y = 2) = \dfrac{\binom{4}{2}\binom{44}{1}}{\binom{48}{3}} = \dfrac{6 \cdot 44}{17296} = \dfrac{264}{17296} = \dfrac{33}{2162}.$

16. $f_Y(y) = \frac{1}{3}(2y+2), \quad 0 < y < 1.$

17. Write $q = 1 - p$.

   (a)

$$M_X(t) = \sum_{k=1}^{\infty} e^{tk}\, q^{k-1} p = p\, e^t \sum_{k=1}^{\infty} (q\, e^t)^{k-1} = \frac{p\, e^t}{1 - q\, e^t}, \qquad q\, e^t < 1.$$

   (b) Differentiating,

$$M_X'(t) = \frac{p\, e^t}{(1 - q\, e^t)^2}, \qquad M_X''(t) = \frac{p\, e^t}{(1 - q\, e^t)^2} + \frac{2p\, q\, e^{2t}}{(1 - q\, e^t)^3}.$$

   At $t = 0$ (using $1 - q = p$):

$$E(X^2) = M_X''(0) = \frac{1}{p} + \frac{2q}{p^2} = \frac{2-p}{p^2}.$$

18. $M_X(t) = \dfrac{\lambda}{\lambda - t}$ for $t < \lambda$; $E(X) = \dfrac{1}{\lambda}$, $E(X^2) = \dfrac{2}{\lambda^2}$.

19. $E(X^2) = \displaystyle\int_0^1 x^2\,(3x^2)\,dx = \frac{3}{5}; \quad E\left(\frac{1}{X}\right) = \int_0^1 \frac{1}{x}\,(3x^2)\,dx = \frac{3}{2}.$

20. $E(X^2) = \displaystyle\int_{-1}^1 x^2\, \tfrac{1}{2}\,dx = \frac{1}{3}; \quad E(|X|) = \int_{-1}^1 |x|\, \tfrac{1}{2}\,dx = \frac{1}{2}.$

21. For $y > 0$,

$$F_Y(y) = P(Y \le y) = P(-\ln X \le y) = P(X \ge e^{-y}) = 1 - e^{-y},$$

   so $f_Y(y) = F_Y'(y) = e^{-y}, \quad y > 0$ (exponential with $\lambda = 1$).

22. Here $F_X(x) = \displaystyle\int_0^x 2t\,dt = x^2$, $0 < x < 1$. For $0 < y < 1$,

$$F_Y(y) = P(X^2 \le y) = P(X \le \sqrt{y}) = F_X(\sqrt{y}) = y,$$

   so $f_Y(y) = F_Y'(y) = 1, \quad 0 < y < 1$ (uniform).

23. (a) $\displaystyle\int_0^1 x(1-x)\,dx = \frac{1}{6}$, so $c = 6$.

   (b) $E(X) = 6\displaystyle\int_0^1 x^2(1-x)\,dx = \frac{1}{2}.$

   (c) $E(X^2) = 6\displaystyle\int_0^1 x^3(1-x)\,dx = \frac{3}{10}$, so $\mathrm{Var}(X) = \dfrac{3}{10} - \dfrac{1}{4} = \dfrac{1}{20}.$