Practice Set: Lecture 09

LOTUS, Mean and Variance, Marginalization, Conditional Densities, Transformations, Covariance, Moment-Generating Functions

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Textbook: Larsen & Marx, *An Introduction to Mathematical Statistics and Its Applications*, Fifth Edition, Chapter 3. All problems are from the text.

Numeric answers are collected at the end. Work every problem before checking.

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1. Let X be binomial with $n = 10$ and $p = \frac{2}{5}$. Find $E(3X - 4)$.
2. Let $f_Y(y) = \sin y$, $0 \le y \le \frac{\pi}{2}$. Find $E(Y)$.
3. Compute the variance of a uniform random variable defined on the unit interval $[0, 1]$.
4. Let $f_Y(y) = 5y^4$, $0 \le y \le 1$. Use $\text{Var}(Y) = E(Y^2) - \mu^2$ to find $\text{Var}(Y)$.
5. Let $f_Y(y) = 3(1 - y)^2$, $0 \le y \le 1$. Find the variance of $W = -5Y + 12$.
6. A set of temperatures recorded in degrees Fahrenheit has standard deviation 15.7°F. The corresponding Celsius temperature is $\frac{5}{9}(Y - 32)$. What is the standard deviation of the equivalent Celsius temperatures?
7. Find the coefficient of skewness $\gamma_1 = E[(Y - \mu)^3]/\sigma^3$ for an exponential random variable with $f_Y(y) = e^{-y}$, $y > 0$.
8. Calculate the coefficient of kurtosis $\gamma_2 = E[(Y - \mu)^4]/\sigma^4 - 3$ for a uniform random variable on the unit interval, $f_Y(y) = 1$, $0 \le y \le 1$.
9. Let X and Y have the joint pmf $p_{X,Y}(x, y) = \dfrac{x + y + xy}{21}$ at the four points $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$. Find the marginal pmf $p_X(x)$ by summing out y.
10. Suppose $f_{X,Y}(x, y) = 6(1 - x - y)$ for x and y in the unit square with $0 \le x + y \le 1$. Find the marginal pdf $f_X(x)$.
11. Two discrete random variables X and Y have joint pdf $p_{X,Y}(x, y) = k(x + y)$ for $x = 1, 2, 3$ and $y = 1, 2, 3$. (a) Find k. (b) Evaluate $p_{Y|x}(1)$ for every x with $p_X(x) > 0$.
12. Suppose $f_{X,Y}(x, y) = \frac{2}{5}(2x + 3y)$, $0 \le x \le 1$, $0 \le y \le 1$. Find (a) $f_X(x)$, (b) $f_{Y|x}(y)$, (c) $P\big(\frac{1}{4} \le Y \le \frac{3}{4} \mid X = \frac{1}{2}\big)$, (d) $E(Y \mid x)$.
13. Let X have pdf $f_X(x) = 3x^2$, $0 \le x \le 1$. Find the pdf of $Y = 2X + 1$.
14. Let Y be exponential with $f_Y(y) = \lambda e^{-\lambda y}$, $y > 0$. Find the pdf of $W = e^{-Y}$.
15. Let X and Y be independent, with $p_X(k) = e^{-\lambda}\lambda^k/k!$ and $p_Y(k) = e^{-\mu}\mu^k/k!$, $k = 0, 1, 2, \ldots$. Find the pdf of $X + Y$, and state whether $X + Y$ belongs to the same family as X and Y.

16. Suppose $f_X(x) = xe^{-x}$, $x \geq 0$, and $f_Y(y) = e^{-y}$, $y \geq 0$, with X and Y independent. Find the pdf of $X + Y$.

17. A contestant takes ten shots with each of two handguns. The final score is 4 times the number of bull's-eyes with the first gun plus 6 times the number with the second. Each shot from the first gun hits the bull's-eye with probability 0.30; each shot from the second, with probability 0.40. Find the expected score.

18. Two fair dice are tossed. Find the expected value of the product of the two faces showing.

19. Two fair dice are thrown. Let X be the number on the first die and Y the larger of the two numbers showing. Find $\mathrm{Cov}(X, Y)$.

20. Let $f_{X,Y}(x, y) = \frac{2}{3}(x + 2y)$, $0 \leq x \leq 1$, $0 \leq y \leq 1$. Find $\mathrm{Var}(X + Y)$.

21. Find the moment-generating function of the discrete random variable X with $p_X(k) = \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)$, $k = 0, 1, 2, \ldots$.

22. Let Y be exponential with $f_Y(y) = \lambda e^{-\lambda y}$, $y > 0$. Use its moment-generating function to find $E(Y^4)$.

23. Find the variance of Y if $M_Y(t) = e^{2t}/(1 - t^2)$.

24. Suppose the moment-generating function of W is

$$M_W(t) = e^{-3+3e^t} \cdot \left(\tfrac{2}{3} + \tfrac{1}{3}e^t\right)^4.$$

Calculate $P(W \leq 1)$.

Answers (numeric only)

1. 8
2. 1
3. $\dfrac{1}{12}$
4. $\dfrac{5}{252}$
5. $\dfrac{15}{16}$
6. 8.72°C
7. $\gamma_1 = 2$
8. $\gamma_2 = -\dfrac{6}{5} = -1.2$
9. $p_X(1) = \dfrac{8}{21}, \quad p_X(2) = \dfrac{13}{21}$
10. $f_X(x) = 3(1 - x)^2$, $0 \leq x \leq 1$

11. (a) $k = \dfrac{1}{36}$; (b) $p_{Y|x}(1) = \dfrac{x+1}{3(x+2)}$: $\frac{2}{9}$ at $x=1$, $\frac{1}{4}$ at $x=2$, $\frac{4}{15}$ at $x=3$

12. (a) $f_X(x) = \dfrac{4x+3}{5}$, $0 \le x \le 1$; (b) $f_{Y|x}(y) = \dfrac{2(2x+3y)}{4x+3}$, $0 \le y \le 1$; (c) $\dfrac{1}{2}$; (d) $E(Y \mid x) = \dfrac{2(x+1)}{4x+3}$

13. $f_Y(y) = \dfrac{3}{8}(y-1)^2, \quad 1 \le y \le 3$

14. $f_W(w) = \lambda\, w^{\lambda-1}, \quad 0 < w < 1$

15. Poisson with parameter $\lambda+\mu$: $p_{X+Y}(k) = e^{-(\lambda+\mu)}(\lambda+\mu)^k/k!$ (same family)

16. $f_{X+Y}(w) = \dfrac{w^2}{2}e^{-w}, \quad w \ge 0$

17. 36

18. $\dfrac{49}{4} = 12.25$

19. $\dfrac{35}{24} \approx 1.458$

20. $\dfrac{5}{36}$

21. $M_X(t) = \dfrac{1}{4-3e^t}, \quad t < \ln \frac{4}{3}$

22. $\dfrac{24}{\lambda^4}$

23. 2

24. $\dfrac{32\, e^{-3}}{27} \approx 0.059$