Problem Set: Lectures 05–07

Bernoulli, Binomial, Geometric, Poisson, Uniform, Exponential

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Textbook: Larsen & Marx, *An Introduction to Mathematical Statistics and Its Applications*, Fifth Edition. All problems are from the text.

Note: No normal-distribution problems appear here; those come next time.

Part A. Core Problems (attempt all ten)

1. Let $X \sim \text{Binomial}(3, 0.2)$. Find the cumulative distribution function $F_X(x)$ and plot its graph.
2. A gunner fires six missiles at an attacking plane; each missile has a 20% chance of being on-target. The plane crashes if two or more missiles find their mark. What is the probability the plane crashes?
3. A teenager passes the driving road test on any given attempt with probability 0.10, attempts being independent. Let X be the number of attempts needed to pass. Write out the pdf $p_X(k)$, and find the average number of attempts required.
4. Flaws in metal sheeting occur at the rate of one per ten square feet. What is the probability that three or more flaws appear in a five-by-eight-foot panel?
5. Fifty spotlights are installed in an outdoor security system. Each bulb burns out at the rate of 1.1 per one hundred hours, so its lifetime is exponentially distributed. What is the expected number of bulbs that fail to last at least seventy-five hours?
6. The amount of calcium Y recovered in an assay is uniformly distributed between 4 and 7 milligrams. Compute $P(Y > 6)$.
7. An urn contains four chips numbered 1 through 4. Two chips are drawn without replacement. Let X be the larger of the two numbers drawn. Find $E(X)$.
8. A random variable Y has pdf $f_Y(y) = 1 - |y|$ for $|y| \le 1$ (and 0 otherwise). Find the cdf $F_Y(y)$ for all real y.
9. In a certain country the disposable income Y of a family has pdf $f_Y(y) = y\,e^{-y}$, $y \ge 0$. Find the cdf $F_Y(y)$.
10. Let $f_Y(y) = \frac{3}{2}\,y^2$, $-1 \le y \le 1$. Verify that f_Y is a legitimate probability density function, and find $P\big(\big|Y - \frac{1}{2}\big| < \frac{1}{4}\big)$.

Part B. Challenging Problems (attempt all five)

11. Three people A, B, C stand in a circle with a faulty revolver and an unlimited supply of bullets. On each turn the person holding the gun fires once at the next living person; the gun discharges (a kill) with probability p and fails otherwise. The gun then passes to the next living person, and firing continues until only one person remains. A shoots first. Let N be the total number of shots fired over the whole game. Find the distribution of N and $E(N)$.

12. Let $U \sim \text{Uniform}(0, 1)$.

    (a) Find the cdf of $Y = -\ln U$, and identify its distribution.

    (b) Let F be a continuous, strictly increasing cdf. Show that $X = F^{-1}(U)$ has cdf F. (Part (a) is the case $F(y) = 1 - e^{-y}$.)

13. Let $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$ be independent, and let $M = \min(X, Y)$.

    (a) Show that $M \sim \text{Exponential}(\lambda + \mu)$, and find $E(M)$.

    (b) Find $P(X < Y)$.

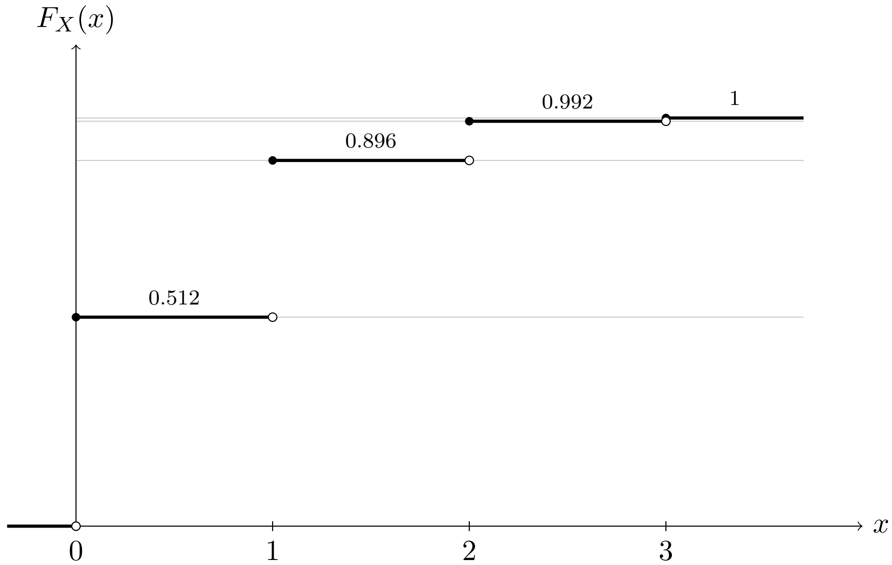
14. Let $X_1, \ldots, X_n$ be independent Uniform$(0, 1)$ random variables, and let $M = \max(X_1, \ldots, X_n)$. Using the cdf method, find the pdf of M and compute $E(M)$.

15. A hen lays $N \sim \text{Poisson}(\lambda)$ eggs in a season; each egg hatches independently with probability p. Let K be the number of chicks. Show that $K \sim \text{Poisson}(\lambda p)$, and find $E(K)$.

Answer Key — Part A

1. $p_X(0) = 0.512$, $p_X(1) = 0.384$, $p_X(2) = 0.096$, $p_X(3) = 0.008$, so the cdf is the right-continuous step function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 0.512, & 0 \le x < 1 \\ 0.896, & 1 \le x < 2 \\ 0.992, & 2 \le x < 3 \\ 1, & x \ge 3. \end{cases}$$



2. $X \sim \text{Bin}(6, 0.2)$. $P(X \geq 2) = 1 - (0.8)^6 - 6(0.2)(0.8)^5 = 1 - 0.6554 \approx 0.345$.

3. $p_X(k) = (0.9)^{k-1}(0.1)$, $k = 1, 2, \ldots$; $\quad E(X) = 1/p = 10$.

4. $\lambda = 40/10 = 4$. $P(X \geq 3) = 1 - e^{-4}(1 + 4 + 8) = 1 - 13e^{-4} \approx 0.762$.

5. Lifetime Y exponential at rate 1.1 per 100 hours; in those units, 75 hours is 0.75. $P(Y < 0.75) = 1 - e^{-1.1(0.75)} = 1 - e^{-0.825} \approx 0.562$. Expected number $= 50(0.562) \approx 28$.

6. $f_Y(y) = \frac{1}{3}$ on $[4, 7]$; $\quad P(Y > 6) = \dfrac{7-6}{3} = \frac{1}{3} \approx 0.333$.

7. The $\dbinom{4}{2} = 6$ pairs are equally likely; the larger number equals 2, 3, 4 for 1, 2, 3 pairs respectively, so $P(X = 2) = \frac{1}{6}$, $P(X = 3) = \frac{2}{6}$, $P(X = 4) = \frac{3}{6}$. $E(X) = \dfrac{2 + 6 + 12}{6} = \dfrac{20}{6} = \dfrac{10}{3} \approx 3.33$.

8. $F_Y(y) = 0$ for $y < -1$; $\quad F_Y(y) = \displaystyle\int_{-1}^{y} (1 + t)\, dt = \frac{(1+y)^2}{2}$ for $-1 \leq y \leq 0$; $\quad F_Y(y) = 1 - \frac{(1-y)^2}{2}$ for $0 \leq y \leq 1$; $\quad F_Y(y) = 1$ for $y > 1$.

9. $F_Y(y) = \displaystyle\int_0^y t\, e^{-t}\, dt = \big[-(t+1)e^{-t}\big]_0^y = 1 - (1 + y)e^{-y}$, $\ y \geq 0$.

10. $f_Y \geq 0$ and $\displaystyle\int_{-1}^{1} \tfrac{3}{2} y^2\, dy = \tfrac{1}{2}[y^3]_{-1}^{1} = 1$, so it is a valid pdf. $P\big(\frac{1}{4} < Y < \frac{3}{4}\big) = \frac{1}{2}[y^3]_{1/4}^{3/4} = \frac{1}{2}\big(\frac{27}{64} - \frac{1}{64}\big) = \frac{13}{64} \approx 0.203$.

Answer Key — Part B

11. Targeting and seating do not matter: each shot is an independent Bernoulli(p) kill (write $q = 1 - p$), and the game stops at the second kill, since three people need two deaths. The

event $N = n$ means exactly one kill among the first $n-1$ shots and a kill on the nth. The single early kill can fall in any of the first $n-1$ positions, so

$$P(N=n) = (n-1)\,q^{n-2}p \cdot p = (n-1)\,p^2 q^{n-2}, \quad n = 2,3,4,\ldots$$

Compute the mean from this distribution directly:

$$E(N) = \sum_{n=2}^{\infty} n\,(n-1)\,p^2 q^{n-2} = p^2 \sum_{n=2}^{\infty} n(n-1)\,q^{n-2}.$$

Differentiating $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ twice gives $\sum_{n=2}^{\infty} n(n-1)q^{n-2} = \frac{2}{(1-q)^3} = \frac{2}{p^3}$, so $E(N) = p^2 \cdot \frac{2}{p^3} = \frac{2}{p}$.

12. (a) For $y \geq 0$, $F_Y(y) = P(-\ln U \leq y) = P(U \geq e^{-y}) = 1 - e^{-y}$; so $Y \sim$ Exponential(1). (b) Since U is uniform and $0 \leq F(x) \leq 1$, $P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x)$. Thus $X = F^{-1}(U)$ has cdf F.

13. (a) $P(M > t) = P(X > t)\,P(Y > t) = e^{-\lambda t}e^{-\mu t} = e^{-(\lambda+\mu)t}$, so $M \sim$ Exponential$(\lambda+\mu)$ and $E(M) = \frac{1}{\lambda+\mu}$. (b) Weight the chance that Y exceeds x by the density of X at x and add over all x: $P(X < Y) = \displaystyle\int_0^{\infty} f_X(x)\,P(Y > x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,e^{-\mu x}\,dx = \frac{\lambda}{\lambda+\mu}$.

14. $F_M(m) = P(\text{all } X_i \leq m) = m^n$, $0 \leq m \leq 1$, so $f_M(m) = n\,m^{n-1}$. $E(M) = \displaystyle\int_0^1 m \cdot n\,m^{n-1}\,dm = \frac{n}{n+1}$.

15. $P(K = k) = \displaystyle\sum_{n=k}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}\binom{n}{k}p^k(1-p)^{n-k} = \frac{e^{-\lambda}(\lambda p)^k}{k!}\sum_{n=k}^{\infty}\frac{(\lambda(1-p))^{n-k}}{(n-k)!} = \frac{e^{-\lambda p}(\lambda p)^k}{k!}$, so $K \sim$ Poisson(λp) and $E(K) = \lambda p$.