Point Estimation: Problem Set

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1. Let T be an estimator of a parameter θ. The bias is

$$b(T) = \mathbb{E}[T] - \theta,$$

and the mean squared error (MSE) is

$$\mathrm{MSE}(T) = \mathbb{E}\big[(T-\theta)^2\big].$$

   (a) Prove that $\mathrm{MSE}(T) = \mathrm{Var}(T) + (b(T))^2$.

   (b) Let $X_1, \ldots, X_n \sim \mathrm{Bernoulli}(p)$. Compare the estimators

$$T_1 = \bar{X}_n, \qquad T_2 = \frac{n\bar{X}_n + 1}{n+2}.$$

   Compute bias, variance, and MSE of each. Which estimator has smaller MSE when n is small? What happens as $n \to \infty$?

2. Suppose $X_1, \ldots, X_n \sim \mathcal{N}(\mu, 1)$ iid. Consider the two estimators of μ:

$$T_1 = \bar{X}_n, \qquad T_2 = c\,\bar{X}_n \quad \text{for some constant } 0 < c < 1.$$

   (a) Compute the bias, variance, and MSE of T_1 and T_2.

   (b) For which values of c (if any) does T_2 have smaller MSE than T_1?

   (c) Draw a plot of the bias versus variance by varying c. You will see a tradeoff which is usually called **Bias-Variance** tradeoff. The bias–variance tradeoff shows that sometimes a slightly biased estimator can give lower overall mean squared error by reducing variance. In economics, the same idea appears when using shrinkage or regularized estimators (e.g. Bayesian priors in macro models), where small bias is accepted for greater stability in finite samples.

3. Let $X_1, \ldots, X_n \sim \mathrm{Poisson}(\lambda)$ iid.

   (a) Find the method-of-moments estimator of λ.

   (b) Find the maximum likelihood estimator of λ.

   (c) Compare the two estimators. Are they equal? Are they unbiased?

   (d) Larsen and Marx: 5.2.4, 5.2.9, 5.5.2.

4. Larsen and Marx: 5.2.11,5.2.14,5.2.17,5.2.21,5.4.1,5.4.9,5.4.11,5.4.13,5.4.19,5.5.6, 5.7.3,