Homework 12 & 13:Hypothesis testing

In this document we will have a lot of practice problems from Larsen and marx on basic hypothesis testing. Then we will solve Neyman Pearson lemma practice problems. Finally I will list bonus problems on this topic from entrance exams for masters programme in India (civil services like IES/ISS, JAM econ, ISI MStat, MSEET, CMI Data Science entrance and so on).

1. L&M 6.2.1, 6.2.2, 6.2.6, 6.3.1, 6.3.7, 6.4.4, 6.4.13, 6.4.19.

2. Denote the power of the most powerful test of level/size α as $MP(\alpha)$. Let $X_1, \ldots, X_n$ be a random sample from p.m.f./p.d.f. f_θ, $\theta \in \Theta = \{\theta_0, \theta_1\}$ where θ_0, θ_1 are known real constants. In each case below, find an MP(α) test ($0 < \alpha < 1$) for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, and determine if it is the unique MP(α) test. Wherever the MP(α) is not unique, find at least two MP(α) tests.

(i) $X_1 \sim N(\theta, \sigma_0^2)$, $\theta_0, \theta_1 \in \mathbb{R}$, $\alpha = 0.90$, σ_0 is a known positive constant.

(ii) $X_1 \sim N(\mu_0, \theta^2)$, $\theta_0, \theta_1 \in (0, \infty)$, $\alpha = 0.95$, μ_0 is a known real constant.

(iii) $X_1 \sim \text{Exp}(\mu_0, \theta)$, $\theta_0, \theta_1 \in \mathbb{R}$, $\alpha = 0.99$, μ_0 is a known real constant.

(iv) $X_1 \sim \text{Exp}(\theta, \sigma_0)$, $\theta_0, \theta_1 \in \mathbb{R}$, $\alpha = 0.95$, σ_0 is a known positive constant.

(v) $X_1 \sim U(0, \theta)$, $\theta_0, \theta_1 \in (0, \infty)$, $\alpha = 0.90$.

(vi) $X_1 \sim U(\theta, \theta + 1)$, $\theta_0, \theta_1 \in \mathbb{R}$, $n \geq 2$, $\alpha = 0.99$.

(vii) X_1 follows discrete uniform distribution on the set $\{1, 2, \ldots, \theta\}$, $\theta_0, \theta_1 \in \{2, 3, \ldots\}$, $\alpha = 0.95$.

(viii) $X_1 \sim \text{Bin}(m, \theta)$, $\theta_0, \theta_1 \in (0, 1)$, $\alpha = 0.95$, m is a known positive integer.

(ix) $X_1 \sim \text{Poisson}(\theta)$, $\theta_0, \theta_1 \in (0, \infty)$, $\alpha = 0.90$.

3. Let $X_1, \ldots, X_5$ be a random sample from $\text{Bin}(2, \theta)$ where $\theta \in \Theta = \{\frac{1}{3}, \frac{2}{3}\}$.

(i) Find an MP(0.95) test for testing $H_0 : \theta = \frac{1}{3}$ vs. $H_1 : \theta = \frac{2}{3}$.

(ii) Find an MP(0.9) test for testing $H_0 : \theta = \frac{2}{3}$ vs. $H_1 : \theta = \frac{1}{3}$.

4. Let $X_1, \ldots, X_5$ be a random sample from $\text{Poisson}(\theta)$ where $\theta \in \Theta = \{1, \frac{3}{2}\}$.

(i) Find an MP(0.95) test for testing $H_0 : \theta = 1$ vs. $H_1 : \theta = \frac{3}{2}$.

(ii) Find an MP(0.9) test for testing $H_0 : \theta = \frac{3}{2}$ vs. $H_1 : \theta = 1$.

Questions from Masters entrance program

1. (IES Stats II 2025, P13) (source) (Easy)

   For testing

   $$H_0 : f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise} \end{cases} \quad \text{against} \quad H_1 : f(x) = \begin{cases} 3x^2, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

   on the basis of a single observation, the power of a most powerful test of size $\alpha = 0.19$ is:

2. (IIT JAM 2023, Statistics, P50) source(Easy)

   Let X_1, X_2 be a random sample from a distribution having a probability density function

   $$f(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

   where $\theta \in (0, \infty)$ is an unknown parameter. For testing the null hypothesis

   $$H_0 : \theta = 1 \quad \text{against} \quad H_1 : \theta \neq 1,$$

   consider a test that rejects H_0 for small observed values of the statistic

   $$W = \frac{X_1 + X_2}{2}.$$

   If the observed values of X_1 and X_2 are 0.25 and 0.75, respectively, then the p-value equals _______________ (round off to two decimal places).

3. (ISI MStat PSB 2019, P1) (source)

   Let Z be a random variable with probability density function

   $$f(z) = \tfrac{1}{2} e^{-|z-\mu|}, \quad z \in \mathbb{R},$$

   with parameter $\mu \in \mathbb{R}$. Suppose we observe $X = \max(0, Z)$.

   (a) Find the constant c such that the test that rejects when $X > c$ has size 0.05 for the null hypothesis $H_0 : \mu = 0$.

   (b) Find the power of this test against the alternative hypothesis $H_1 : \mu = 2$.

4. (IIT JAM 2023, Statistics) source (Medium)

   Let X_1, X_2 be a random sample from a $U(0, \theta)$ distribution, where $\theta > 0$ is an unknown parameter. For testing the null hypothesis

   $$H_0 : \theta \in (0, 1] \cup [2, \infty) \quad \text{against} \quad H_1 : \theta \in (1, 2),$$

   consider the critical region

   $$R = \left\{ (x_1, x_2) \in \mathbb{R} \times \mathbb{R} : \frac{5}{4} < \max\{x_1, x_2\} < \frac{7}{4} \right\}.$$

   Then, the size of the critical region equals __________.

5. (ISI MStat PSB 2024, P9) (source) (Medium-Hard)

   To test whether the heights of siblings are correlated, a researcher devised the following plan: she identified a random sample of n families with at least two adult male children. For the ith family, suppose that X_i and Y_i are the heights of the first and second male child, respectively. Assume that $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent bivariate normal random vectors with parameters $(\mu, \mu, \sigma^2, \sigma^2, \rho)$, where μ and σ^2 are known from previous studies. She is interested in testing the null hypothesis

   $$H_0 : \rho = 0 \quad \text{against} \quad H_1 : \rho = 0.5.$$

   Unfortunately, due to a mistake in the questionnaire, she was only able to observe (U_i, V_i) for each i, where

   $$U_i = \max(X_i, Y_i) \quad \text{and} \quad V_i = \min(X_i, Y_i).$$

   (a) Based on the observed sample, obtain the test statistic corresponding to the most powerful test of H_0 against H_1.

   (b) Find a critical value so that the size of the test converges to 0.05 as $n \to \infty$.

6. (ISI MStat PSB 2020, P8) (source) (Hard)

   Assume that $X_1, \ldots, X_n$ is a random sample from $N(\mu, 1)$, with $\mu \in \mathbb{R}$. We want to test

   $$H_0 : \mu = 0 \quad \text{against} \quad H_1 : \mu = 1.$$

   For a fixed integer $m \in \{1, \ldots, n\}$, the following statistics are defined:

   $$T_1 = \frac{X_1 + \cdots + X_m}{m}, \quad T_2 = \frac{X_2 + \cdots + X_{m+1}}{m}, \quad \ldots, \quad T_{n-m+1} = \frac{X_{n-m+1} + \cdots + X_n}{m}.$$

   Fix $\alpha \in (0, 1)$. Consider the test

   $$\text{Reject } H_0 \text{ if } \max\{T_i : 1 \le i \le n - m + 1\} > c_{m,\alpha}.$$

   Find a choice of $c_{m,\alpha} \in \mathbb{R}$ in terms of the standard normal distribution function Φ that ensures that the size of the test is at most α. [Hint: Use P(intersection of A_i) $\le$ sum of $P(A_i)$.]