Statistics: Internals 2 Supplementary

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Duration: 90 minutes **Maximum scorable Marks:** 20

OPEN NOTEBOOK EXAM. PRINT, PHOTOCOPY MATERIALS NOT ALLOWED.
Attempt all questions. Show clear reasoning. Partial credit will be given for significant steps.

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1. (1 mark each) Answer briefly. Explanations must be precise, not essay-style.

(a) Fill in the blanks: Weak law of _____ numbers states that the sample mean of random variables sampled from a distribution converges to the expectation of the distribution, as the number of samples tends to infinity.

(b) Suppose we sampled 5 numbers x_1, x_2, x_3, x_4, x_5 from Poisson(λ) where λ is unknown. Is the function $\lambda(x_1 + x_2 + x_3 + x_4 + x_5)$ a statistic? Explain.

(c) Explain the use of the Rao–Blackwell theorem.

(d) Write down the probability density function of the multivariate normal distribution. You may assume the mean vector is μ and the covariance matrix is Σ.

(e) A test for a rare disease is 99% accurate. A person tests positive and claims there's a 99% chance they are infected. What's wrong with their claim?

2. (2 marks) Suppose a coin is to be tossed n times for the purpose of estimating p, where $p = P(\text{heads})$. How large must n be to guarantee that the length of the 99% confidence interval for p will be less than 0.02? (You may use $z_{0.995} = 2.5758$.)

3. (2 marks) Use the method of moments to estimate θ in the pdf

$$f_Y(y;\theta) = (\theta^2 + \theta)\, y^{\theta-1}(1-y), \qquad 0 \le y \le 1.$$

Assume that a random sample of size n has been collected. [Hint: If you identify the distribution,you can name it and use any fact relating to it to solve the problem.]

4. (3 marks) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the pdf

$$f_Y(y;\theta) = \frac{1}{(r-1)!\theta^r} y^{r-1} e^{-y/\theta}, \quad y > 0$$

Show that $\hat{\theta} = \frac{1}{r}\bar{Y}$ is a minimum-variance unbiased estimator for θ.

5. (3 marks) Suppose that Y is a gamma random variable with parameters r and θ, and the prior is also gamma with parameters s and μ. Show that the posterior pdf is gamma with parameters $r + s$ and $y + \mu$.

6. (3 marks) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a uniform pdf defined over the interval $[0, 1]$. Find the pdf and the expectation of the i-th order statistic $Y_{[i]}$, for a given natural number i between 1 and n.

7. (6 marks) Let $X_1, \ldots, X_n$ be i.i.d. samples from a truncated exponential distribution on $[0, 1]$ with pdf

$$f(x; \lambda) = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda}}, \quad 0 \leq x \leq 1, \, \lambda > 0.$$

(a) Show that $T = \sum_{i=1}^{n} X_i$ is a sufficient statistic for λ.

(b) Derive the likelihood equation for the MLE $\hat{\lambda}$ and show that it satisfies an implicit equation in T of the form $T = g(\hat{\lambda})$.

(c) (Hard) If the sample mean is 0.55, then what is the value of $\hat{\lambda}$?