MADRAS SCHOOL OF ECONOMICS

UNDERGRADUATE PROGRAMME IN ECONOMICS (HONOURS) [2022-25]

SEMESTER 5 [JULY – NOVEMBER, 2024]

REGULAR EXAMINATION, NOVEMBER 2024

Course Name: Stochastic Process **Course Code:** DE13

Duration: 2 Hours **Maximum Marks: 60**

Instructions: *For part A write short answers (most preferably a single word,/single number/single sentence). For part B, writing relevant formulae and quoting correct definitions helps me give you part marks. For questions with qualitative answers, creative writing is prohibhited.*

Part A: Answer all questions (1 mark x 10 questions = 10 marks)

1. Define a discrete time markov chain.
2. If $N(t)$ is a Poisson process with rate λ, what is the distribution of the interarrival times?
3. If X is distributed Exponential(λ), compute $\Pr\{X > \lambda\}$.
4. Define a Wiener process.
5. Is it possible that the mean number of visits to a given state in an irreducible DTMC is finite? Explain
6. Let $N(t)$ be a Poisson process, then what is the value of $\mathbb{E}(N(t))$?
7. Let P_{ij} be the transition probability matrix of the embedded DTMC of a homogenous CTMC, then what is the value of $1 - P_{11}$?
8. Can the sojourn time of a pure jump CTMC be a constant random variable? Explain.
9. If you had to model the number of insurance claims in a given period of time, which stochastic process will you use?
10. An arbitrary cumulative distribution function F is given to you. What is the range of values F can take?

Part B: Answer any five. (5 mark x 10 questions = 50 marks)

11. Answer each subpart in not more than 100 words (be succinct and highlight stochastic concepts):
    (a) [3 marks] Explain the notion of stopping time with an example. Where is it used in theory?
    (b) [4 marks] Sketch the derivation of Ito's lemma and explain an application of the lemma in Finance.
    (c) [3 marks] Outline the Cramer-Lundberg model and the theory of ruin emphasising the Poisson process.

12. Answer the following questions:
    (a) [6 marks] Derive the distribution of a Wiener process over an interval as a scaled limit of a symmetric random walk.
    (b) [4 marks] Let $W(t)$ be a Wiener process, and $0 \leq s < t$. Find the conditional PDF of $W(t)$ given $W(s) = a$.

13. Let $N(t)$ be a Poisson process with rate λ
    (a) [6 marks] Let T_2 be the second arrival time. Compute the conditional density

    $$f_{T_2|N(t)}(x|N(t) = 3)$$

    Check that the density is a pdf.
    (b) [4 marks] Compute the expectation

    $$\mathbb{E}(T_2|N(t) = 3).$$

14. Let S_n denote the number of heads in n fair coin tosses. Then
    (a) [3 marks] Show that S_n is binomially distributed. What is the mean and variance of S_n?
    (b) [3 marks] Show that S_n has the Markov property.
    (c) [2 marks] Show that S_n has the independent increment property.
    (d) [2 marks] Find a function $f(n)$ such that the process $S_n - f(n)$ is a martingale.

15. Let the rate of transition matrix R of a two state Markov chain (with states $S = \{0, 1\}$) be

    $$R = \begin{bmatrix} -\lambda & \mu \\ \lambda & -\mu \end{bmatrix}.$$

    (a) [8 marks] Solve the system of equations by computing $P(t) = e^{Rt}$.
    (b) [2 marks] Write your guess of the parameters of soujorn time for state 0 and 1. No derivations/justifications required.

16. Consider the state graph of a markov chain shown in Figure 1.
    (a) [3 marks] If you assume the self arrow at 1 and the one-step chance for state 7 to jump to state 5 is 20% chance,what is the probability assigned to the other unmarked arrows?
    (b) [4 marks] List the communicating classes of the markov chain. Indicate which classes are transient and which ones are positive recurrent.
    (c) [3 marks] What is the mean number of visits to state 4 if the markov chain starts in state 3?

17. Let $P(t)$ be the price of a stock and suppose $W(t) := \log P(t)$ is a Wiener process.
    (a) [2 marks] Show that increment of $W(t)$ over an interval $[t, t + T]$ is related to the return on investment.
    (b) [6 marks] Show that $W(t)^2 - t$ is a martingale.
    (c) [2 marks] Show that the distribution of $W(-t)$ is also normal.

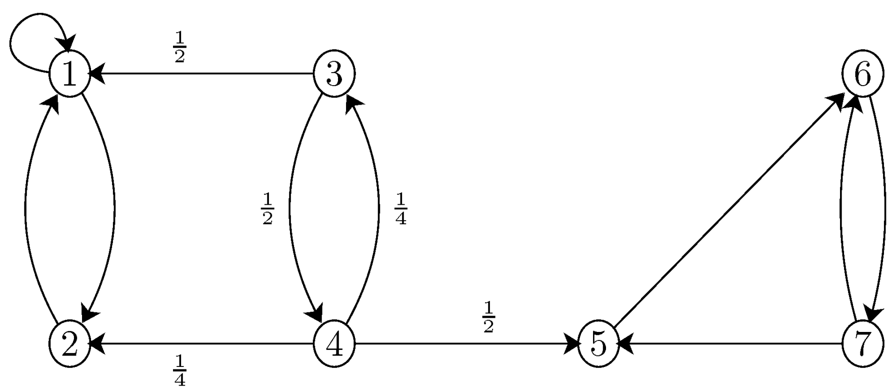

Figure 1: