MADRAS SCHOOL OF ECONOMICS

UNDERGRADUATE PROGRAMME IN ECONOMICS (HONOURS) [2022-25]

SEMESTER 6 [JANUARY – MAY, 2025]

REGULAR EXAMINATION, MAY 2025

Course Name: Real Analysis **Course Code:**GE03

Duration: 2 Hours **Maximum Marks: 60**

Instructions: *For part A write short answers (justify using any result proved in class). For part B, writing relevant formulae and quoting correct definitions helps me give you part marks.*

Part A: Answer all questions (1 mark x 10 questions = 10 marks)

1. State the least upper bound axiom of real numbers.
2. Define a convergent series of real numbers.
3. Write down a bijection between natural numbers ending with 0 and $\mathbb{N}$. No proof required.
4. What are all the limit points of the set of first 100 natural numbers? Quote a theorem to justify your answer.
5. Give an example of sequence which has only two distinct subsequential limits. No proof required.
6. Define lim sup (upper limit/limit superior) of a real sequence.
7. Let $X = \mathbb{R}^2$. Define a distance function as

$$d(x, y) = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

   Mention one axiom of metric space that fails.

8. Explain why the following series is convergent?

$$\sum_{n=0}^{\infty} (0.5)^{n^2}$$

9. Let $x_{n+1} = \dfrac{5}{x_n} + \dfrac{x_n}{2}$ for $n > 1$ with $x_1 = 3$. Compute

$$\lim_{n \to \infty} x_n$$

10. Give an example of a bounded sequence which is not convergent.

Part B: Answer any five. (10 mark x 5 questions = 50 marks)

11. Squeeze theorem

    (a) [1 marks] State the Squeeze theorem for sequences.

    (b) [3 marks] Prove the Squeeze theorem using limsup and liminf properties. Clearly state the properties used to prove.

    (c) [2 marks] State the precise theorems relating to algebra of limits of sequences.

    (d) [4 marks] Use the previous three parts to compute the limit of

    $$\lim_{n\to\infty} \frac{1-(0.5)^n}{n^{\frac{1}{n}}} + \frac{(-1)^n \cos n \sin n}{e^n(n^2+1)}.$$

    All the steps must be argued rigorously and you may quote limit of any special sequence we have learnt.

12. Series

    (a) [2 marks] Define a convergent series of real numbers. When is a series of non-negative terms convergent?

    (b) [4 marks] Define e^x using a power series and prove the power series is convergent for $0 < x \leq 1$. [Hint:Squeeze it between 0 and a geometric series.] Conclude $e^x > 1 + x$ for $0 < x < 1$.

    (c) [4 marks] Using previous part or otherwise, prove that the series

    $$\sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n}\right)$$

    is convergent.

13. Closed and compact sets

    (a) [5 marks] If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection is nonempty, then the total intersection is nonempty:

    $$\bigcap_\alpha K_\alpha \neq \emptyset.$$

    (b) [2 marks] Define compact sets and state Heine-Borel theorem.

    (c) [3 marks] Give an example of compact set with countable number of elements and prove your claims.

14. Cardinality of $\mathbb{R}$

    (a) [5 marks] Prove that the set of real numbers in the interval $(0, 1)$ is uncountable.

    (b) [3 marks] Prove that an infinite subset of a countably infinite set is countable.

    (c) [2 marks] Use parts (a) and (b) to prove that $\mathbb{R}$ is uncountable.

15. Let l^∞ denote the set of all bounded sequences of real numbers. If $x = \{x_n\}_{n=1}^\infty$ and $y = \{y_n\}_{n=1}^\infty$ are points in l^∞, define

    $$d(x, y) = \sup_{1\leq n<\infty} |x_n - y_n|.$$

    (a) [6 marks] Prove that d is a metric on l^∞.

    (b) [2 marks] If $x = \{1 + \frac{1}{n}\}_{n=1}^\infty$ and $y = \{2 - \frac{1}{n}\}_{n=1}^\infty$, then what is $d(x, y)$?

    (c) [2 marks] Consider the constant sequence $x = \{1, 1, 1, 1, \cdots\}$. Describe the open neighbourhood of x with radius $\frac{1}{2}$ with respect to the metric d.

16. Answer the following questions.

    (a) [4 marks] In a metric space X, prove that every convergent sequence is a Cauchy sequence.

    (b) [2 mark] Explain the principle of mathematical induction.

    (c) [4 marks] Prove that $11^n - 4^n$ is divisible by 7 when n is a positive integer.

17. Answer the following questions:

(a) [2 marks] Use Cauchy condensation theorem to show

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is not convergent.

(b) [4 marks] Let E be a subset of a metric space. Prove that the set of all limit points of E is a closed set.

(c) [2 marks] Prove that $\mathbb{R}$ is not compact with respect to the discrete metric.

(d) [2 marks] Consider the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$. Does it converge in $\mathbb{R}$ with the discrete metric?