Sequences and Series

Practice problems on sup/inf

1. Show that
$$\mathbf{sup}\{x \in \mathbb{Q} : x > 0, x^2 < 2\} = \sqrt{2}.$$

2. Let $A \subseteq R$ be a nonempty set. Define
$$-A = \{-x : x \in A\}.$$
Show that
$$sup(-A) = -infA,$$
$$inf(-A) = -supA.$$

3. Find the sup and inf of the following sets
$$A = \left\{2(-1)^{n+1} + (-1)^{\frac{n(n+1)}{2}}\left(2 + \frac{3}{n}\right)\right\}.$$

Sequences

1. If a sequence (a_n) is monotonically increasing, then prove that
$$\lim_{n\to\infty} a_n = sup\{a_n : n \in N\}$$
Conclude that if it was monotonically decreasing then the limit would be the infimum.[Hint: Use Problem 2 from previous section.]

2. Show that a monotone bounded sequence converges.[Hint: Use compactness in the range of the sequence.]

3. State and prove squeeze theorem.

4. Discuss the existence of limits for the following sequences and compute the limits when it exists:

   (a) $x_n = \frac{n}{2^n}$ for $n \in \mathbb{N}$.

   (b) $x_n = \frac{n!}{2^{n^2}}$ for $n \in \mathbb{N}$.

   (c) $x_n = \sin n$ for $n \in \mathbb{N}$.

   (d) $x_n = (1^2 + 2^2 + \cdots + n^2)^{\frac{1}{n}}$ for $n \in \mathbb{N}$.

   (e) $x_n = \frac{n}{2^{\sqrt{n}}}$ for $n \in \mathbb{N}$.

Subsequences and limsup/liminf

1. [IMPORTANT] Let $\limsup_{n\to\infty} a_n = L$. Show that for any $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that
$$n \geq N \implies a_n < L + \epsilon.$$

2. Prove that a necessary and sufficient condition for the convergence of a sequence (a_n) is that both the limit inferior and the limit superior are finite and

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$$

3. Let $\{a_n\}$ be a sequence whose subsequences $(a_{2k}), (a_{2k+1})$ and (a_{3k}) are convergent.
   (a) Prove that the sequence (a_n) is convergent.
   (b) Does the convergence of any two of these subsequences imply the convergence of the sequence (a_n)?

4. Prove

$$\liminf_{n\to\infty} a_n = \frac{1}{\limsup_{n\to\infty} a_n}$$

5. Recall that a sequence (a_n) is a function $a : \mathbb{N} \to \mathbb{R}$. Let E denote the set of subsequential limits of the sequence (a_n).

   Prove or disprove: The set of limit points of the range of the function a is E.

6. Compute limsup and liminf of all the sequences:
   (a) $a_n = 1$
   (b) $a_n = \frac{1}{n}$
   (c) $a_n = \left(4^{(-1)^n} + 2\right)^{\frac{1}{n}}$
   (d) $a_n = \frac{2n^2}{5} - \left[\frac{2n^2}{5}\right]$ where $[x]$ is the greatest integer function.

7. Prove the following inequalities:

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

   Give an example where equality fails.