Problem sheet 3 - Topology of metric spaces.

Definition 1. A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function called a *metric* (or *distance function*), satisfying the following properties for all $x, y, z \in X$:
1. **Non-negativity**: $d(x, y) \geq 0$,$d(x, y) = 0$ if and only if $x = y$.
2. **Symmetry**: $d(x, y) = d(y, x)$.
3. **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$.

If (X, d) is a metric space, note that for any subset $Y \subseteq X$, the pair (Y, d) is also a metric space. Usually I (Srikanth) might call it a submetric space.

Exercise 1. In order to draw pictures, show the following distance functions are metrics:

1. $X = \mathbb{R}$ and for $x, y \in X$, $d(x, y) = |x - y|$.
2. $X = \mathbb{R}^n$ and for $x, y \in X$, $d(x, y) = ||x - y||$, where $||v|| = \sqrt{< v, v >}$ is the norm of the vector v.[Hint: Look up Cauchy Schwarz Inequality.]
3. $X = \mathbb{R}^n$ and for $x, y \in X$, $d(x, y) = max\{|x_k - y_k| : 1 \leq k \leq n\}$, where v_k is the kth coordinate of the vector v.
4. (Discrete metric) For any set X, define $d(x, y) = 1$ if x is not equal to y and if $x = y$, define $d(x, y) = 0$.

Exercise 2. Let (X, d) be a metric space, define another distance function

$$\delta(x, y) := \frac{d(x, y)}{1 + d(x, y)};$$

Show that δ is a metric on X.

Economics students often run into with time series problems. Real Analysis is no different. For now we state a fact that we will revisit later.

Fact 1. *Let* $1 \leq p < \infty$ *and let*

$$\ell_p = \left\{ (y_t)_{t=1}^{\infty} : y_t \in \mathbb{R} \ or \ \mathbb{C} \ and \ \sum_{t=1}^{\infty} |y_t|^p < \infty \right\}$$

The elements of ℓ_1 are called absolutely summable sequences *and play an important role in time series but usually in time series we run the time index from $-\infty$ to $+\infty$.Let $(y_t) \in \ell_p$, define*

$$||y|| = \left(\sum_{t=1}^{\infty} |y_t|^p \right)^{\frac{1}{p}}$$

then it is a fact that the distance function $d(x, y) = ||x - y||$ is a metric on ℓ_p.

1 Open and closed sets

1. Define a neighborhood, limit point of a set, an open set and a closed set.
2. Show that any neighborhood is open.
3. Show that any neighborhood of a limit point of a set contains infinitely many points of the set. Prove the converse as well.
4. Show that complement of open set is closed and vice versa.
5. Prove that an arbitrary union of open sets is open and finite intersection of open sets is open.Conclude that arbitrary intersection of closed sets is closed and finite union of closed sets is closed.
6. Define the closure of a set E and show that it is the "smallest closed set" containing E.
7. If (Y, d) is a submetric space of (X, d), then if E is an open subset of Y, we say E is open relative to Y.
   (a) Show an example of a set that is open relative to Y but not X.
   (b) Show that E is open relative to Y iff $E = Y \cap G$ for some set G open in X.

2 Homework

Submit 1,2,3,4 by Tuesday 18th Feb 2025.

1. Construct a bounded set of real numbers with exactly 3 limit points.
2. List all the open and closed sets of the discrete metric space.
3. Let E' be the set of limit points of E.Prove that E' is closed.
4. Let E° be the set of interior points of a set E in a metric space. Prove the following:
   (a) E° is open.

(b) (Universal property) If V is any open set of the metric space such that $V \subseteq E$, then $V \subseteq E^\circ$.

(c) E is open iff $E^\circ = E$.

(d) For any open set E, $\overline{E}^\circ = E$? [Hint" What happens when $E = \mathbb{Q}$?]

5. *A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have$x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$. Prove that every separable metric space has a countable base. [Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X.]