HOMEWORK - 2

1. Polynomial functions

(1) Prove that the set of all $\mathcal{P}_d$ polynomial functions of degree d with integer coefficients is countable.

(2) Prove that the set of all polynomial functions $\mathcal{P} = \cup_{d=1}^{\infty} \mathcal{P}_d$ with integer coefficients is countable.

(3) Show that the set of all polynomial functions with rational coefficients is countable.

(4) A real number x is called an *algebraic number* if x is the root of a polynomial function f with rational coefficients, i.e. $f(x) = 0$. A *transcendental number* is a real number which is not an algebraic number.
   (a) Prove that the set of algebraic numbers is countable.
   (b) Prove that the set of transcendental numbers is uncountable.
   [You can assume that a polynomial of degree n has at most n roots.]

2. Countable sets

(1) We have proved that every open interval (a, b) contains a rational number. Show that (a, b) contains infinitely many rationals.

(2) Prove that any infinite set contains a countable set. [*Hint* - Use the definition. A set A is infinite if for each natural number n, there is a subset of A with n elements]

(3) Suppose a set A is infinite and $x \in A$, show that there is a bijection from A to $A - \{x\}$.

(4) Construct bijective functions between the following sets.
   (a) $[a, b] \to [0, 1]$
   (b) $[0, 1] \to [a, b]$
   (c) $(0, 1) \to (a, b)$
   (d) $(a, b) \to (0, 1)$
   (e) $(0, 1) \to (-\infty, \infty)$

(5) Is the set of all functions from the set of all natural numbers $\mathbb{N}$ to the set $\{0, 1\}$ countable or uncountable?

(6) If the set of all functions from the set $\{0, 1\}$ to $\mathbb{N}$ countable or uncountable?

(7) Give a bijection from $\mathbb{N}$ to the set of all odd integers greater than 13.

3. Some basics

(1) You are given subsets A and B of the set of real numbers $\mathbb{R}$. Draw diagrams to represent the Cartesian product $A \times B$ as a subset of $\mathbb{R}^2$ in the plane.
   (a) $A = [1, 2] \cup [3, 4]$, $B = \{1, 2\}$
   (b) $A = \{1, 2, 3\}$ and $B = [1, 2]$

(2) Consider the function $f(x) = \frac{1}{x^2}$ for $x \in \mathbb{R} \setminus \{0\}$.
   (a) What is the image of the set $[1, 2]$ under f?
   (b) What is the inverse image of the set $[1, 4]$ under f?

(3) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$.
   (a) Determine the number of one to one maps from $B \to A$.
   (b) Determine the number of onto maps from $A \to B$.