PROBLEM SET - 1

1. Mathematical Induction

(1) Prove that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ for all positive integers n.

(2) Prove that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all positive integers n.

(3) Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all positive integers n.

(4) Guess a formula for $1 + 3 + 5 + \cdots + (2n - 1)$ and prove it by induction.

(5) Prove that $11^n - 4^n$ is divisible by 7 when n is a positive integer.

2. Real number system

(6) Prove that $\sqrt{3}$ is not a rational number (try proving without using the theorem about rational roots).

(7) Prove that $\sqrt{2 + \sqrt{2}}$ is not a rational number.

(8) Show that $\sqrt{\dfrac{4 - 2\sqrt{3}}{7}}$ is not a rational number.

(9) Show that $\sqrt{4 + 2\sqrt{3}} - \sqrt{3}$ is actually a rational number.

(10) Show that there is no rational number r such that $2^r = 3$.

(11) Are the following sets bounded above / below? What is the supremum and infimum in each case? Justify your statements.
(a) $\cap_{n=1}^{\infty}[-\frac{1}{n}, 1 + \frac{1}{n}]$
(b) $\cap_{n=1}^{\infty}(1 - \frac{1}{n}, 1 + \frac{1}{n})$
(c) $\{\sin(\frac{n\pi}{3}) : n \in \mathbb{N}\}$
(d) $\{x \in \mathbb{R} \,:\, x^3 < 8\}$
(e) $\{\frac{m}{n} : m, n \in \mathbb{N}, m < n\}$

(12) Let S be a non-empty bounded subset of $\mathbb{R}$. Prove that $\inf S \leq \sup S$.

(13) Let S and T be a non-empty bounded subsets of $\mathbb{R}$. Let $S \subseteq T$. Prove that $\inf T \leq \inf S \leq \sup S \leq \sup T$.

(14) Let S and T be non-empty bounded subset of $\mathbb{R}$. Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

(15) Let $a, b \in \mathbb{R}$ with $a < b$. Show that there are infinitely many rationals between a and b.

(16) Let $a, b \in \mathbb{R}$. Suppose $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then show that $a \leq b$.

(17) Let $a, b \in \mathbb{R}$ with $a < b$. Consider the set $T = [a, b] \cap \mathbb{Q}$. What is the supremum of T? Prove it.

3. Functions

(18) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Find $f(A)$, $f(B)$, $f(A \cap B)$, $f(A) \cap f(B)$, $f(A \cup B)$ and $f(A) \cup f(B)$ where A and B are the closed intervals $A = [0, 2]$ and $B = [1, 4]$. Find two sets A, B where $f(A \cap B) \neq f(A) \cap f(B)$.

(19) Let $f(x) = \log(x)$ for $x \in (0, \infty)$. What is the range of f? Let $A = [0, 1]$ and $B = [1, 2]$. Find $f^{-1}(A)$, $f^{-1}(B)$ and $f^{-1}(A \cap B)$.

(20) Let $f(x) = \sin(x)$ for $x \in (-\infty, \infty)$.
  (a) What is $f^{-1}(1)$?
  (b) Find $f([0, \pi/6])$, $f([\pi/6, \pi/2])$, $f([0, \pi/2])$ and $f([5\pi/6, \pi])$.

(21) Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f(x) = (\cos(x), \sin(x))$. What is the range of f?

(22) Let $A = \{1, 2, \cdots, n\}$ and $B = \{0, 1\}$. How many functions are there which map A to B? How many of these are onto functions?