

Real Analysis: Internals-2

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Rules of the exam

1. 90 minutes, 25 marks paper. Maximum marks you can score is 20.
2. Anyone in possession of mobile/smartwatch or any electronic device capable of connecting to a network will be given a zero.
3. You are allowed to use definitions, theorems and formulae proved in class. Any other methods will not score marks allowed unless you prove all the relevant claims implicit in your method.
4. **Please write an explicit counterexample to disprove a claim.**
5. All the questions follow notations used in class. You may use a calculator.

Problems

1. [3 marks] Prove that any convergence sequence of real numbers is bounded.
2. [3 marks] Define a convergent sequence. Given two convergent sequences of real numbers $\lim_{n \rightarrow \infty} s_n = s$, $\lim_{n \rightarrow \infty} t_n = t$, prove that

$$\lim_{n \rightarrow \infty} s_n t_n = st.$$

3. [4 marks] Write the answer to each of the following question:

- (a) Write the negation of the statement: For every $a \in A$, it is true that $a^2 \in B$.
- (b) Prove or disprove: For any two sets A, B

$$A - (B - A) = A - B.$$

- (c) List all the open and closed sets of the discrete metric space.

4. [4 marks] Compact sets:

- (a) Define a compact set.
- (b) Show that every infinite subset of a compact set has a limit point.

5. [3 marks] State whether each of the following sets are compact in the given metric space. Justify your answer using a theorem proved in class.

- (a) $[0, 1] \cup \{3\}$ in \mathbb{R} with standard metric.
- (b) $\{(x, y) \in \mathbb{R}^2 | x^2 \leq y\}$ in \mathbb{R}^2 with the standard metric.

6. [3 marks] If A, B are compact subsets of a metric space, then prove that $A \cup B$ is also compact.

7. [5 marks] Let E denote the set of subsequential limits of a sequence (x_n) .

- (a) [2 marks] Construct a sequence (x_n) such that E is infinite. You have to prove your claim.
- (b) Construct a sequence (x_n) such that E is uncountable. You have to prove your claim. [Note if you solve part b alone, you get 5 marks since it also solves part a.]