MADRAS SCHOOL OF ECONOMICS

UNDERGRADUATE PROGRAMME IN ECONOMICS (HONOURS) [2024-27]

SEMESTER FOUR [JANUARY – APRIL, 2026]

SUPPLEMENTARY EXAMINATION, JUNE 2026

Course Name: Differential Equations **Course Code:**GE06

Duration: 2 Hours **Maximum Marks: 60**

Instructions: *For part A write short answers (most preferably LESS than a couple of sentences). For part B, writing relevant formulae and quoting correct definitions helps me give you part marks.*

Part A: Answer all questions without justification (1 mark x 10 questions = 10 marks)

1. Write an example of a separable first order differential equation that is not linear.
2. State the superposition principle for solutions of a homogeneous linear differential equation.
3. Define the Wronskian of two differentiable functions $y_1(t)$ and $y_2(t)$.
4. What is the Laplace transform of $te^{-2t}u(t)$ (here $u(t)$ is the step function)?
5. State any two of the Dirichlet conditions for the convergence of a Fourier series.
6. Draw a phase portrait of a saddle point.
7. Write the formula for the particular solution y_p using variation of parameters for $y'' + p(t)y' + q(t)y = g(t)$, assuming y_1, y_2 are known independent solutions of the homogeneous equation.
8. What does the Fourier series of a piecewise smooth function converge to at a point of discontinuity?
9. For a 2×2 linear system $\mathbf{x}' = A\mathbf{x}$, under what condition on the eigenvalues of A is the origin a center?
10. Write down a second order, constant coefficient linear differential equation whose Wronskian is constant.

Part B: Answer any five. (10 mark x 5 questions = 50 marks)

11. Consider the second order linear differential equation

$$y'' + p(t)y' + q(t)y = 0$$

where $p(t)$ and $q(t)$ are continuous on an interval I.

   (a) [5 marks] Prove: if y_1 and y_2 are any two solutions, then their Wronskian satisfies

$$W(t) = C \exp\left(-\int p(t)\,dt\right).$$

   for some constant C.

   (b) [5 marks] Using previous part, prove that y_1 and y_2 are linearly independent on I if and only if $W(t) \neq 0$ for all $t \in I$.

12. Solve the following differential equations and write down the complete solution:

   (a) [4 marks] $\dfrac{dy}{dx} = \dfrac{x+2y}{2x+y}$

   (b) [6 marks] $y'' - 2y' + y = \dfrac{e^t}{t^2+1}$ using variation of parameters.

13. Consider the differential equation

$$(1-x^2)y'' - 2xy' + 6y = 0.$$

   (a) [2 marks] Verify that $x = 0$ is an ordinary point.

   (b) [6 marks] Assume a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Derive the recurrence relation for the coefficients a_n.

   (c) [2 marks] Find a polynomial solution of degree 2.

14. Let $f(x) = x(\pi - x)$ for $0 < x < \pi$.

   (a) [6 marks] Find the Fourier sine series of $f(x)$ on the interval $(0, \pi)$.

   (b) [4 marks] Using the Dirichlet theorem, state precisely what the series converges to at $x = 0$, $x = \pi/2$, and $x = \pi$.

15. Answer the following questions on Laplace transform $\mathcal{L}$:

   (a) [4 marks] Show that

$$\mathcal{L}\{t\sin(2t)\} = \frac{4s}{(s^2+4)^2}.$$

   (b) [6 marks] Solve the initial value problem

$$y'' + 4y = 4\cos(2t), \quad y(0) = 0, \quad y'(0) = 0.$$

16. Solve the partial differential equation (wave equation):

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2,\ t > 0,$$

$$u(0,t) = 0, \quad u(2,t) = 0, \quad t > 0,$$

$$u(x,0) = \sin(\pi x) - \frac{1}{2}\sin(2\pi x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < 2.$$

17. Consider the nonlinear system

$$\frac{dx}{dt} = x - xy, \qquad \frac{dy}{dt} = -y + xy.$$

   (a) [2 marks] Find all critical points of the system.

   (b) [4 marks] For the critical point at the origin, find the linearization and classify its type and stability.

   (c) [4 marks] For the critical point at $(1,1)$, find the linearization, compute its eigenvalues, and classify the critical point. State what the Hartman–Grobman theorem guarantees about the local phase portrait of the nonlinear system near this point.