GE06 ODE: Problems on PDEs, Systems of ODEs

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Instructions: Show all working. For phase portraits, indicate the direction of trajectories as $t \to \infty$. For PDEs, verify that your solution satisfies both the equation and boundary/initial conditions.

1. Consider the linear system
$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$
   (a) Find the eigenvalues and eigenvectors of the coefficient matrix.
   (b) Write down the general solution.
   (c) Classify the critical point at the origin. Is it stable, unstable, or asymptotically stable?
   (d) Sketch the phase portrait, clearly indicating the eigenvector directions and the direction of motion along trajectories.

2. Consider the linear system
$$\frac{dx}{dt} = -x + 2y, \qquad \frac{dy}{dt} = -2x - y.$$
   (a) Write this system in matrix form $\mathbf{x}' = A\mathbf{x}$ and find the eigenvalues of A.
   (b) Find the general solution in real form.
   (c) Classify the critical point at the origin. Determine whether trajectories spiral inward, outward, or form closed curves.
   (d) Sketch the phase portrait.

3. Consider an economy where output Y adjusts toward goods market equilibrium and the interest rate r adjusts toward money market equilibrium:
$$\frac{dY}{dt} = \alpha(C + I - Y), \qquad \frac{dr}{dt} = \beta(L - M)$$
where $C = c_0 + c_1 Y$ is consumption, $I = i_0 - i_1 r$ is investment, $L = \ell_0 + \ell_1 Y - \ell_2 r$ is money demand, and M is fixed money supply. All parameters are positive with $c_1 < 1$.
   (a) Show that this system can be written as $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ where $\mathbf{x} = (Y, r)^T$. Find A and $\mathbf{b}$.
   (b) Find the unique equilibrium (Y^*, r^*).
   (c) Show that the equilibrium is asymptotically stable if $\alpha(1 - c_1) + \beta\ell_2 > 0$ and $\alpha\beta[(1 - c_1)\ell_2 + i_1\ell_1] > 0$. Interpret these conditions economically.
   (d) Under what parameter conditions would the equilibrium be a stable spiral? What would this mean for the adjustment path following a monetary shock?

4. In Goodwin's classical model of distributive conflict (paper was shared in GCR), let u denote the labor share of output and v the employment rate. Their dynamics are:

$$\frac{du}{dt} = u(-\gamma + \rho v), \qquad \frac{dv}{dt} = v(\delta - \sigma u)$$

where $\gamma, \rho, \delta, \sigma > 0$. High employment strengthens worker bargaining power, raising the labor share; a high labor share squeezes profits and reduces capital accumulation, lowering employment.

   (a) Find all critical points in the region $u \geq 0$, $v \geq 0$.

   (b) For the interior critical point, compute the Jacobian and show that the eigenvalues are purely imaginary. What type of critical point is this?

   (c) Does the Hartman–Grobman theorem apply? Explain.

   (d) Show that $H(u, v) = \sigma u - \delta \ln u + \rho v - \gamma \ln v$ is constant along trajectories. What does this imply about the phase portrait?

   (e) Sketch the phase portrait in the region $u > 0$, $v > 0$. Interpret the cycles economically: what happens to wages and employment over one complete cycle?

5. Consider the nonlinear system

$$\frac{dx}{dt} = x(2 - x - y), \qquad \frac{dy}{dt} = y(3 - 2x - y).$$

   (a) Find all critical points of the system.

   (b) For each critical point, compute the Jacobian matrix and find its eigenvalues.

   (c) Classify each critical point as a node, saddle, spiral, or center. Determine stability where applicable.

   (d) Which of your critical points are hyperbolic? For any non-hyperbolic critical point, explain why the HG theorem does not apply.

   (e) Sketch the phase portrait in the first quadrant $x \geq 0$, $y \geq 0$, showing all critical points and representative trajectories.

6. Show that if the matrix A associated to a two dimensional linear system is of rank one then the nullspace of A is a **critical line** of singularities. Further the column space of A is invariant under the flow. [An invariant set is a set that contains the whole trajectory if the initial point was in the set.]

7. Consider the system

$$x' = x + y + bxy, \qquad y' = x + y - bxy$$

where $b \neq 0$ is a parameter.

   (a) Show that the linearization at the origin has eigenvalues 2 and 0. Why does the Hartman–Grobman theorem not apply?

   (b) Introduce new coordinates $u = x + y$ and $v = x - y$. Show that the system becomes

$$u' = 2u, \qquad v' = \frac{b}{2}(u^2 - v^2).$$

(c) Verify that if $u(0) = 0$, then $u(t) = 0$ for all t. That is, the line $x + y = 0$ is invariant under the flow.

(d) On the invariant line $x + y = 0$, set $y = -x$ and show that the dynamics reduce to

$$x' = -bx^2.$$

Solve this equation explicitly. For $b > 0$, which direction do solutions move along this line?

(e) Solve $u' = 2u$ explicitly. For a trajectory starting at (x_0, y_0) with $x_0 + y_0 > 0$, show that $x(t) + y(t) \to +\infty$ as $t \to \infty$. What happens when $x_0 + y_0 < 0$?

Non-examinable remark: The v-equation is Riccati and can be linearized to a modified Bessel equation of order zero. From the asymptotics of Bessel functions, $v/u \to \pm 1$ as $t \to \infty$. In original coordinates this means trajectories bend toward the coordinate axes as they escape, not along the eigenvector $(1, 1)$. Use the portrait plotter online to appreciate this remark.

(f) Sketch the phase portrait near the origin for $b > 0$, combining your answers to (d) and (e).

Hint for PDE problems below: Use separation of variables $u(x, y) = X(x)Y(y)$. You will obtain a Sturm–Liouville problem in x, which determines the eigenvalues.

8. Solve the heat equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0,$$

subject to the boundary conditions

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0,$$

and the initial condition

$$u(x, 0) = 3\sin(2\pi x) - \sin(5\pi x), \quad 0 < x < 1.$$

9. Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2,\ t > 0,$$

subject to the boundary conditions

$$u(0, t) = 0, \quad u(2, t) = 0, \quad t > 0,$$

and the initial conditions

$$u(x, 0) = \sin(\pi x) - \frac{1}{2}\sin(2\pi x), \quad \frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 < x < 2.$$

10. Solve Laplace's equation on a rectangular domain:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi,\ 0 < y < 1,$$

subject to the boundary conditions

$$u(0, y) = 0, \quad u(\pi, y) = 0, \quad 0 < y < 1,$$

$$u(x, 0) = 0, \quad u(x, 1) = \sin(2x), \quad 0 < x < \pi.$$