Ordinary Differential Equations: Mock Test

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90 minutes, 48 marks, Open Notes

1. [8 marks] Solve the following problems. You may use the result of part (a) for part (b) if needed.

   (a) Compute: $\int \cosh(x)\sin(ax)\,dx$ where a is a real parameter. [4 marks]

   (b) Find the Fourier series $F(x)$ for

   $$f(x) = \begin{cases} \cosh(x) & x \in (-11, 0) \\ -\cosh(x) & x \in (0, 11) \end{cases}$$

   on the interval $[-11, 11]$. [4 marks]

2. [8 marks] Consider the 2π-periodic function on $[-\pi, \pi)$:

   $$f(x) = \begin{cases} -1 & -\pi \le x < -\frac{\pi}{2} \\ x & -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x < \pi \end{cases}$$

   (a) Verify that the function satisfies the Dirichlet conditions for Fourier series. Identify all points of discontinuity and compute the pointwise convergence value of the Fourier series at $x = \frac{\pi}{2}$. [4 marks]

   (b) Sketch the periodic extension on $[-2\pi, 2\pi]$. Which Dirichlet condition must be checked at the boundary points $x = \pm\frac{\pi}{2}$? Find all points where the Gibbs phenomenon occurs, compute the jump values and justify your answer. [4 marks]

3. [12 marks] Consider the following differential equation

   $$y''(t) + 4y'(t) + 4y(t) = \begin{cases} e^{-t} & 0 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

   with initial conditions $y(0) = 1$ and $y'(0) = 0$.

   (a) Take the Laplace transform of both sides and compute $Y(s) = \mathcal{L}\{y(t)\}$. [4 marks]

   (b) Solve for $y(t)$ by computing the inverse Laplace transform. [8 marks]

4. [12 marks] Let $V = \{f : [0,1] \to \mathbb{R} \mid f, f', f'' \text{ are continuous on } [0,1],\ f(0) = f(1) = 0\}$.

   Define the inner product and operator:

$$\langle f, g \rangle = \int_0^1 f(x)g(x)e^x \, dx, \quad T(f) = f'' + f'.$$

   (a) Show that T is self-adjoint with respect to the given inner product, i.e., $\langle Tf, g \rangle = \langle f, Tg \rangle$ for all $f, g \in V$. [6 marks]

   (b) Solve the eigenvalue problem: $T(f) = \lambda f$ with $f(0) = f(1) = 0$. Find all eigenvalues λ and corresponding non-zero eigenfunctions. [6 marks]

5. [8 marks] Consider the Sturm–Liouville problem:

$$(x^2 y')' + \lambda y = 0, \quad x \in (1, e),$$

   with boundary conditions $y(1) = y(e) = 0$.

   (a) Solve the differential equation and find all eigenvalues λ_n and corresponding eigenfunctions $y_n(x)$. [5 marks]

   (b) Show that eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $w(x) = 1$:

$$\int_1^e y_m(x) y_n(x) \, dx = 0 \quad \text{for } m \neq n.$$

   [3 marks]