Internals-2
GE06: Ordinary Differential Equations

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Total Marks: 48 **Time:** 120 minutes

Instructions: Attempt all questions. You can use hand written notes. Show all working; Justifications for steps should come from theorems.

1. (4 marks) Answer **either** Part (a) or Part (b). Indicate your choice clearly at the top of your answer.

   (a) **Essay.**

   Write a note on Gibbs phenomenon. Explain the phenomenon in detail and state the facts known about it. No derivations required. Graphical explanations will be appreciated. [Not more than 200 words]

   (b) **Derivation**

   Derive the Laplace transform below from first principles using the definition.

$$\mathcal{L}[te^t].$$

   No properties of Laplace transform are allowed. Carefully discuss the region for s where the laplace transform $F(s)$ will converge.

2. [6 marks] Consider the 2π-periodic function on $[-\pi, \pi)$:

$$f(x) = \begin{cases} -1 & -\pi \leq x < -\frac{\pi}{2} \\ x & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x < \pi \end{cases}$$

   (a) Verify that the function satisfies the Dirichlet conditions for Fourier series. Identify all points of discontinuity and compute the pointwise convergence value of the Fourier series at $x = \frac{\pi}{2}$. [3 marks]

   (b) Sketch the periodic extension on $[-2\pi, 2\pi]$. Which Dirichlet condition must be checked at the boundary points $x = \pm\frac{\pi}{2}$? Find all points where the Gibbs phenomenon occurs, compute the jump values and justify your answer. [3 marks]

3. [8 marks] Let $f : \mathbb{R} \to \mathbb{R}$ be defined on one period by

$$f(x) = \begin{cases} x, & 0 \leq x \leq \pi, \\ 2\pi - x, & \pi < x \leq 2\pi, \end{cases}$$

   and let $g : [-\pi, \pi] \to \mathbb{R}$ be defined by $g(x) = |x|$.

(a) Compute the Fourier series of f. Does the series equal the value of the function at all points? Justify. [6 marks]

(b) Compute the Fourier series of g. [2 marks]

4. [10 marks] Consider the differential equation

$$y''(t) + 2y'(t) + 5y(t) = \begin{cases} e^{-t} & 0 \le t < \pi, \\ 0 & t \ge \pi, \end{cases}$$

with initial conditions $y(0) = 1$ and $y'(0) = -1$.

(a) Take the Laplace transform of both sides and compute $Y(s) = \mathcal{L}\{y(t)\}$. [4 marks]

(b) Hence find $y(t)$ by inverse Laplace transform only! You may use any property or formulae from the table of Laplace transform to solve it. Any other method will not be entertained. [6 marks]

5. [15 marks] Let V be the space of functions $f : (-1, 1) \to \mathbb{R}$ that are twice continuously differentiable and remain bounded as $x \to \pm 1$. Define the inner product and operator

$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)\, g(x)}{\sqrt{1-x^2}}\, dx, \qquad T(f) = (1-x^2)\, f'' - x\, f'.$$

(a) Show that $T(f) = \sqrt{1-x^2}\,\dfrac{d}{dx}\left[\sqrt{1-x^2}\, f'\right]$. [3 marks]

(b) Hence show that T is self-adjoint with respect to the given inner product. Explain why boundedness of f, g at ± 1 is enough for T to be self-adjoint. [6 marks]

(c) Find all eigenvalues and corresponding non-zero eigenfunctions of $T(f) = \lambda f$ that are bounded at ± 1. You have to derive the answer; quoting the correct answer will not net you most of the marks. [Hint: Substitute $x = \cos\theta$ to simplify the terms.] [6 marks]

6. [5 marks] Let $F(p) = \mathcal{L}\{f(t)\}$. You are given the following identity:

$$\int_0^\infty \frac{f(t)}{t}\, dt = \int_0^\infty F(p)\, dp.$$

Evaluate $\displaystyle\int_0^\infty \frac{e^{-t} - e^{-2t}}{t}\, dt$. [5 marks]

[Remark: Don't attempt to integrate this directly using Calc-1 methods, you will fail!]